Brief Sketches of Post-Calculus Courses

The bulletin descriptions of courses often give you little idea about the topics of the courses, unless you happen to already know the many technical terms that the descriptions use. To address this gap, we offer brief, largely self-contained sketches that are written to give you a glimpse of our upper-level offerings. It may be especially useful to read these sketches around registration time, when you are deciding which courses to take next semester, or when planning for future courses.

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Consider the following objects and operations.

- For polynomials with coefficients in the real numbers, \( \mathbb{R} \), you know how to add two polynomials, and also how to multiply a polynomial by a real number. Both operations yield another polynomial. You can do the same with polynomials over the rational numbers, \( \mathbb{Q} \), if you multiply them only by rational numbers. You can also replace \( \mathbb{R} \) by the complex numbers, \( \mathbb{C} \).

- For functions \( f : \mathbb{R} \to \mathbb{R} \) (that is, the domain and range is \( \mathbb{R} \)), you know how to add two functions, and to multiply a function by a real number, and the result is another real-valued function of a real variable. As above, you can replace \( \mathbb{R} \) by \( \mathbb{Q} \) or \( \mathbb{C} \). Also, for \( \mathbb{R} \), you can focus just on the functions that are differentiable everywhere, or those that are integrable on closed intervals.

- You might know about adding two elements in \( \mathbb{R}^3 \), and multiplying by a real numbers, using the operations \((a, b, c) + (r, s, t) = (a + r, b + s, c + t)\) and \(k(a, b, c) = (ka, kb, kc)\).

Certain properties are common to all such examples: for instance, addition is commutative and associative. The same structure (a set of objects with operations of addition of these objects and multiplication of them by numbers, all obeying certain rules) appears in many important contexts in addition to those above. The first focus of attention in linear algebra, vector spaces, abstracts these examples. In a vector space, we have a set (whose elements we call vectors) and a field (such as \( \mathbb{R} \) or \( \mathbb{Q} \) or \( \mathbb{C} \)) and operations of addition of two vectors and multiplication of a vector by a scalar in the field (e.g., by a number in \( \mathbb{R} \)), satisfying eight familiar rules.

Using the operations of addition of polynomials and multiplication by numbers, every polynomial can be written in exactly one way in terms of the polynomials in the set \( \{1, x, x^2, \ldots\} \). Likewise, each vector in \( \mathbb{R}^3 \) can be written in exactly one way as a sum of scalar multiples of the vectors in the set \( \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \); in particular, \((a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)\). These are examples of bases of vector spaces. Each vector in a vector space can be described uniquely by its coordinates relative to a given basis. A vector space has many different bases, but, in a given vector space, all bases have the same number of vectors; that number is the dimension of the vector space. Bases, dimension, and related notions occupy much of the first part of Math 2185.

Two formulas that you know from calculus,

\[
\frac{d}{dx}(f(x) + k \cdot g(x)) = f'(x) + k \cdot g'(x) \quad \text{and} \quad \int f(x) + k \cdot g(x) \, dx = \int f(x) \, dx + k \int g(x) \, dx,
\]

show that some very important operations preserve sums and scalar multiples. Generalizing this, the second focus of attention in linear algebra is functions \( T \) from one vector space to another, over the same field, that preserve the operations in the sense that \( T(u + av) = T(u) + aT(v) \) for all vectors \( u \) and \( v \) in the domain, and elements \( a \) of the field. Such functions are called linear transformations.

Matrices make many appearances in linear algebra, starting with the basic problem of solving systems of linear equations (which can be cast as matrix equations). For a linear transformation \( T : V \to U \) between finite-dimensional vector spaces, matrices provide efficient ways to obtain the coordinates of the image \( T(v) \) in a given basis of \( U \) from the coordinates of the vector \( v \) in a given basis of \( V \). The connections between linear transformations and matrices leads to a rich theory of decomposing linear transformations \( T : V \to V \) into simpler pieces.

This course and Math 2184 have much in common (so credit may not be earned for both), but the perspective is different, with Math 2184 emphasizing calculation, and Math 2185 emphasizing theory, proof, and an abstract point of view. Among the things we gain with the more general setting of Math 2185 are (a) efficiency: one proof in the general setting replaces separate proofs in each of the particular examples, and (b) insight: we find out which properties of interest in a given example hold in general and which are particular to that example.

Prerequisites: Math 1231 and Math 2971 (Math 2971 and 2185 may be taken simultaneously).
The great advantages of organizing our mathematical knowledge into chains of deductions, going from definitions and explicitly-stated assumptions (axioms) to deep and powerful conclusions, have been widely recognized for more than two and a half millennia. The resulting structure solidifies our knowledge and makes it easier to digest, appreciate, and remember what we know, to share mathematics with each new generation, and to observe patterns that suggest directions for further development. Mathematical proofs serve as the mortar that makes this structure rock-solid, and they provide the explanations of why everything works. Proofs and proof-related skills are the focus of Math 2971.

In many ways, Math 2971 is a gateway — a passage to new horizons.

• It gives an introduction to how to read, understand, devise, and write mathematical proofs. In particular, it carefully analyzes and illustrates the process of coming up with the ideas that go into constructing proofs.

• Besides treating basic logic and proof techniques (e.g., quantifiers, proof by contradiction, induction, using the contrapositive), it covers many fundamental structures and concepts that you will be assumed to be familiar with in upper-level courses (e.g., basic set theory, equivalence relations, set partitions, cardinality). Once you have mastered these topics in Math 2971, you can focus on the really new ideas in upper-level courses.

• This course expands your understanding of what mathematics is. While there is not sufficient time to include surveys of numerous parts of math, the course uses examples drawn from a variety of branches of math (with enough background provided to make the examples accessible) and so gives you a glimpse of some of the fascinating worlds that lie beyond the introductory-level courses.

• Math 2971 fosters skills to tackle non-routine problems.

• As a WID course, it provides you with practice and feedback that promote clear and precise writing in the style used by professional mathematicians.

For many reasons, it is desirable and highly recommended to take Math 2971 as early as possible, either at the same time as Math 1232 (Calculus II) or just after that course. In addition to being a crucial prerequisite for many of our upper-level courses, this course exposes you to what lies beyond the subjects that you are already familiar with and gives you a different perspective on math; as a result, it can help you decide which of our three tracks for the mathematics major (pure, applied, and interdisciplinary) may work best for you.

The target audience for this course is declared or prospective mathematics majors and minors. Everyone who enrolls in this course should be eager to grapple with challenging mathematical ideas and problems. This is a WID course, so expect a fair bit of writing. (Those who are not considering mathematics as a major or minor should talk with the instructor before enrolling in this course since those looking just to fulfill a WID requirement could find this course too deep.)

Prerequisite or co-requisite: Math 1232.

Recommended follow-up courses for practicing your proof-writing and problem-solving skills: Math 2185 or any 3000-level, theoretically-oriented class.

Math 2185, Linear Algebra For Math Majors
Math 3120, Elementary Number Theory
Math 3125, Linear Algebra II
Math 3257, Complex Variables
Math 3613, Combinatorics
Math 3632, Graph Theory
Math 3710, Mathematical Logic
Math 3720, Axiomatic Set Theory
Math 3730, Computability Theory
Math 3740, Computational Complexity
Math 3806, Topology
Math 3848, Differential Geometry

(Many students benefit from taking one or more of these courses before taking 4000-level courses.)
Number theory treats properties of the positive integers. “Elementary” in elementary number theory reflects the use of relatively basic techniques (not the level of difficulty), in contrast to, for instance, algebraic and analytic number theory, which use higher-level algebra or analysis.

You might already have seen some beginning topics from Math 3120 in Math 2971, such as the division algorithm (a basic result that is behind many topics in number theory), the fundamental theorem of arithmetic (every positive integer factors uniquely into primes), the equivalence relation of congruence modulo an integer \( n \), and the fact that there are infinitely many primes. Below we describe some other topics that elementary number theory treats.

A strengthening of the statement that there are infinitely many primes is that the series

\[
\sum_{\text{primes } p} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots
\]

diverges. This series is a variation on the harmonic series, but thinned out to just the reciprocals of the primes. The divergence is proven in elementary number theory. Such results give us some idea of the relative density of primes among the integers. For instance, it shows that primes are denser than squares since \( \sum_{n \geq 1} \frac{1}{n^2} \) converges.

Primes have a wealth of interesting properties. To cite just two examples, a prime \( p \) always divides \( 1 \cdot 2 \cdot 3 \cdots (p-1) + 1 \), that is, \( p|(p-1)! + 1 \). For instance, \( 4! + 1 = 25 \), which is divisible by 5. Also, for any prime \( p \) and any integer \( a \) that is not a multiple of \( p \), the number \( a^{p-1} - 1 \) is always divisible by \( p \). For instance, \( 7|53^6 - 1 \). Why do such properties always hold? Such properties and their consequences are explored in elementary number theory.

The second property of primes just cited has a counterpart that replaces the prime \( p \) by any positive integer \( n \), and the exponent \( p-1 \) by \( \phi(n) \), which is the number of integers not exceeding \( n \) that have no prime factors in common with \( n \); the integer \( n \) divides \( a^{\phi(n)} - 1 \) when \( a \) and \( n \) have no common factors. These ideas play key roles in one of the major applications of number theory to cryptography. Number theory is the basis of the currently best methods for encrypting data, such as your e-mail.

In elementary number theory, we show how to generate all Pythagorean triples, that is, all triples \( a, b, c \) of integers (such as 3, 4, 5, or 5, 12, 13) that satisfy the Pythagorean relation \( a^2 + b^2 = c^2 \). This brings up the topic of writing numbers as sums of squares. Notice that the primes 3, 7, and 11 cannot be written as the sums of two squares, while

\[
2 = 1^2 + 1^2, \quad 5 = 2^2 + 1^2, \quad 13 = 3^2 + 2^2, \quad 17 = 4^2 + 1^2.
\]

Which primes are sums of two squares? Elementary number theory yields the answer, and from that, we deduce which positive integers \( n \) are the sums of two squares.

Observe that we can write \( 6 \) as the sum of three squares, \( 6 = 2^2 + 1^2 + 1^2 \), while 7 requires four squares, \( 7 = 2^2 + 1^2 + 1^2 + 1^2 \). Do some integers require more squares? No! In elementary number theory, we show that one never needs more than four squares: for any positive integer \( n \), the equation \( n = x_1^2 + x_2^2 + x_3^2 + x_4^2 \) has an integer solution.

Number theory has a wealth of tantalizing open problems, ranging from classical problems (e.g., the twin prime conjecture (there are infinitely many pairs of primes of the form \( p, p+2 \), such as 3 and 5, or 5 and 7) and Goldbach’s conjecture (every even integer 4 and greater is the sum of two primes; e.g., 24 = 13 + 11)) to recent problems. See http://www.openproblemgarden.org/category/number_theory_0 and https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics.

For a fuller, yet still brief, description, see https://en.wikipedia.org/wiki/Number_theory.

Prerequisites: Math 2971.
Like its prerequisite Math 2185, this course is a theory-oriented study of Linear Algebra that deepens students’ understanding of vector spaces and linear transformation in particular, and rigorous mathematics in general. The choice of topics is somewhat flexible, but the following describes some key areas that are typically covered.

You know that if $V$ is a vector space of dimension $m$ and $T: V \rightarrow V$ is a linear transformation, then choosing a basis for $V$ allows us to write down an $m \times m$ matrix $A_T$ that captures the action of $T$. But what is that action? Is the effect of $T$ to rotate, flip, stretch, shrink, some combination of these, or something else? A large portion of the material in Math 3125 can be seen as providing ways to answer this question. For instance, if $W$ is a subspace of $V$ such that every vector in $W$ is mapped by $T$ to a vector in $W$ (possibly the same vector, but not necessarily), then we call $W$ a $T$-invariant subspace. The simplest example of this is an eigenspace; if $T(v) = \lambda v$ for some scalar $\lambda$, then the linear span of $v$ is a 1-dimensional $T$-invariant subspace. Understanding the relationships between the $T$-invariant subspaces allows us to choose a basis for $V$ such that the structure of the matrix $A_T$ reveals important information about the action of $T$ on $V$. For example, if $V$ has a basis consisting of eigenvectors of $T$, then $A_T$ will be diagonal. Jordan normal form, a topic in Math 3125, provides an approach in terms of invariant subspaces for many cases when $V$ has no basis of only eigenvectors.

Another important concept that highlights linear transformations enjoying a particular type of action is that of inner product and, more generally, bilinear form. Let $F$ denote the field of scalars for $V$. A function $\varphi: V \times V \rightarrow F$ is called bilinear if it is linear in each coordinate; for vectors $v, w$ in $V$ we can view $\varphi(v, w)$ as providing some sort of measurement of the relationship between $v$ and $w$. With some additional properties, $\varphi$ will be an inner product, and thus will provide us with a notion of distance and angle in $V$. Given this we can ask what effect a linear transformation $T: V \rightarrow V$ will have on the lengths of vectors and the angle between them. Linear transformations that leave lengths and angles unchanged are called orthogonal when $F = \mathbb{R}$ (real numbers) and unitary when $F = \mathbb{C}$ (complex numbers). Not surprisingly, orthogonality and unitarity tie in with many special properties. In Math 3125 you will learn about this, and in particular you will learn how orthogonal and unitary transformations can be decomposed in revealing ways.

In Math 2185 you were exposed to the idea that linear transformations can be viewed as elements of a vector space. Of course, this means that the reverse is true as well! Given a vector $v$ in a vector space $V$ over the scalar-field $F$ we can use $v$ to define a function $V \rightarrow F$ by the rule $w \mapsto v^T w$ for all $w \in V$. More generally, we can define a vector space $V^*$ called the dual of $V$ consisting of all linear functions $V \rightarrow F$. When $V$ is finite dimensional, all elements of $V^*$ arise in the way described above, but the story is more complicated for infinite dimensional vector spaces. The vector space dual is a useful tool in a variety of areas of advanced mathematics and science, as it provides a bridge between vectors and linear transformations. For example, in the bra-ket notation in physics a ket $|w\rangle$ is a vector whereas a bra $\langle v|$ is a dual vector; the symbol $\langle v|w\rangle$ then denotes applying the function $\langle v|$ to the vector $|w\rangle$.

Linear Algebra is a very rich area with many uses within mathematics and many applications to science. A strong grasp of the material in Math 3125 can offer great benefits when taking other courses and when pursuing a mathematical career.

Prerequisites: Math 2971 and 2185.
Complex analysis deals with the calculus of complex-valued functions $f : \mathbb{C} \to \mathbb{C}$. A real-valued function of a real variable maps an interval to another interval, but a complex-valued function of a complex variable maps a two-dimensional region to a two-dimensional region. To make sense of the derivative of $f$, it is natural to try to extend the definition of the derivative of a real-valued function to $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, i.e.,

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}.$$  

Evaluating the limit first along the real and then the imaginary axis, we find

$$f'(z) = u_x + iv_x \quad \text{and} \quad f'(z) = -iu_x + v_y,$$

respectively. In order for $f'(z)$ to be well-defined, we are led to the conditions:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \quad (1)$$

Equations (1) are called the Cauchy-Riemann equations; they are foundational in complex analysis and lead to some surprising behavior of complex analytic functions.

A mapping $z = f(\zeta)$ with $f(\zeta)$ analytic and $f'(\zeta) \neq 0$ preserves the angle between curves. Such a mapping is called a conformal mapping and can be used to solve problems in fluid flow, electrostatics, and other fields. Can you find an analytic function that conformally maps the interior of an ellipse to the unit disk? Or the upper-half complex plane to the exterior of the unit circle? Check out Complex Analysis and Conformal Mappings by Peter J. Olver or Wolfram MathWorld.

The notion of line integral $\int_C f(z)\,dz$ extends to a complex curve $C \subset \mathbb{C}$ and a complex-valued function $f(z)$. The connection between analyticity and the complex integral is the content of the Cauchy-Goursat theorem: If $C$ is a simple closed curve whose derivative is continuous except at a finite number of points and $C$ is inside a simply connected region $D$ in which $f$ is analytic, then $\int_C f(z)\,dz = 0$. A converse of Cauchy’s Theorem (Morera’s Theorem) states that if $\int_C f(z)\,dz = 0$ for a continuous $f$ and every closed curve in $R$, then $f$ is analytic in $R$. Cauchy’s theorem can be used to prove the fundamental theorem of algebra, i.e., every polynomial of positive degree has at least one zero.

Another surprising result is presented by the Cauchy integral formula, which asserts that the values of an analytic function on the boundary of a disk determine its value at every interior point. If $f(z)$ is analytic in a simply connected region $D$ and $C$ is a simple closed positively-oriented curve in $D$ and $z$ is a point inside $C$, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\,dw.$$  

In fact, using complex integration and the Cauchy integral formula one can show that the values of an analytic function in its domain are completely determined by its values on an arbitrarily short curve in the domain.

The two-dimensional character of complex-valued functions is one of the reasons that complex analysis is so effective in two-dimensional problems in mathematical physics (hydrodynamics, thermodynamics, and quantum mechanics). In addition, complex analysis is useful in many branches of mathematics: algebraic geometry, number theory, combinatorics, partial differential equations, and applied mathematics.

Prerequisites: Math 2184 or 2185, Math 2233, and 2971.
MATH 3342, ORDINARY DIFFERENTIAL EQUATIONS

(This description is being written.)

Prerequisites: Math 2184 or 2185, and Math 2233.
Almost every theory in the physical world is built around a partial differential equation. There are the Navier-Stokes equation in fluid mechanics, the Maxwell equation in electromagnetism, the Schrödinger equation in quantum mechanics, and the Einstein equation in general relativity. In Finance the Black-Scholes equation evaluates the price of stock options; in Biology the Gierer-Meinhardt equation describes pattern formation and morphogenesis in cell development. Within Mathematics, analytic functions studied in Complex Analysis are just solutions of the Cauchy-Riemann equation; Soap films studied in Geometry are solutions of the minimal surface equation; in 2003 Grigori Perelman solved the Poincaré conjecture, a fundamental problem in Topology, using a partial differential equation called the Ricci flow.

Here is one of the equations you will encounter in this class. Consider a heat conducting rod of length \( L \) and we want to know its temperature. The temperature, denoted by \( u \), is not a number but a field. It varies from point to point on the rod and also changes in time, so \( u \) is a function of the space variable \( x \) in the interval \((0, L)\) and the time variable \( t > 0 \). Thermodynamics tells us that the temperature function \( u(x, t) \) satisfies the following heat equation:

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2}.
\]

This equation is a partial differential equation because it involves some partial derivatives of the unknown variable \( u \). To solve this problem, we also need to understand how the rod interacts with the external environment. As an interval in the real line, the rod meets the outside world at the two end points: \( x = 0 \) and \( x = L \). It is possible that the temperature is maintained at the fixed level, say \( A \) degrees at \( x = 0 \) and \( B \) degrees at \( x = L \). Then one imposes the boundary condition

\[
u(0, t) = A, \quad u(L, t) = B, \quad \text{for all} \quad t > 0.
\]

It is also possible that the rod is insulated from the environment and there is no heat flow at the end points. Then one has a different boundary condition

\[
\frac{\partial u(0, t)}{\partial x} = 0, \quad \frac{\partial u(L, t)}{\partial x} = 0, \quad \text{for all} \quad t > 0.
\]

Finally, we need to know the temperature distribution at the initial time, i.e.

\[
u(x, 0) = \phi(x), \quad \text{for all} \quad x \in (0, L)
\]

where \( \phi \) is a given function called the initial condition. We will learn how to solve the heat equation for the unknown function \( u \) when proper boundary and initial conditions are provided. One of the main methods in this course is called separation of variables. It relies on the theory of trigonometric series in Advanced Calculus.

In the second half of the semester, we will consider bodies more complex than a rod. Say you have a heat conducting solid body, called \( D \), in space. How does the temperature of the body evolve? Now we have an unknown function \( u(x, y, z, t) \) of \((x, y, z)\) in \( D \), which is a region in the three dimensional Euclidean space, and of time \( t > 0 \). The heat equation becomes

\[
u(x, y, z, t) = \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2}
\]

We must develop different approaches for different shapes of \( D \).

Other equations studied in this course include the Laplace equation which is used to study gravity and static electric fields, the wave equation for sound, water, and electromagnetic waves, and the Shrödinger equation for quantum harmonic oscillators.

Prerequisites: Math 2233, Math 2184 or 2185, and Math 3342.

You must know how to use the divergence theorem, how to integrate by parts in multiple integrals, and how to find eigenvalues in ordinary differential equations.
Math 3359, Introduction to Mathematical Modeling

Mathematical modeling is the process of describing in mathematical terms some phenomenon or idea that is not initially perceived or described from a mathematical perspective. For example, one may try to develop equations to describe how some physical events are related, or a formal algorithm to capture how some decision is arrived at. Once this movement from the empirical world to the mathematical world has been carried out, one can bring a host of mathematical techniques to bear in an attempt to explain and/or predict aspects of the empirical world.

For instance, suppose a population of organisms is such that in an environment that provided unlimited resources, the population grows continuously, and at all times proportionally to its size $u(t)$. Writing $a$ for the constant of proportionality, we can describe the population size by the differential equation $\frac{du(t)}{dt} = au(t)$. Now add the extra hypothesis that the organisms live in an environment that can only support up to a certain population size, say $K$, and that due to intra-species competition, the growth rate is inversely proportional to how close $u(t)$ is to $K$. Under these assumptions, the population size can be described by the following more elaborate differential equation:

$$\frac{du(t)}{dt} = au(t) \left(1 - \frac{u(t)}{K}\right).$$

This is called the “logistic equation,” and it has a very nice solution $u(t)$. But now suppose that the organisms in question have seasonal fertility, so that growth occurs in jumps. Keeping all the other assumptions the same while adjusting to use a sequence $u_0, u_1, u_2, \ldots$ rather than a continuous function $u(t)$, we can model this growth with a difference equation $u_{n+1} = u_n + au_n \left(1 - \frac{u_n}{K}\right)$. This is called the “discrete logistic equation,” and it does not have a very nice solution; in fact, it leads to what is called “chaotic behavior.” Constructing, studying, and contrasting examples of this kind is a key aspect of this course.

Mathematical modeling is a big field, and enjoys a variety of approaches, including well-known statistical and machine-learning techniques. In line with the above example, however, this course focuses primarily on “modeling from first principles.” This means that one begins by identifying key principles believed to be at play in the context to be modeled, expresses these principles in terms appropriate to some area of mathematics, then derives the logical implications, thereby developing an appropriate mathematical theory. Some elementary techniques of data-analysis are usually discussed, although they are not a primary focus.

The types of mathematical models considered in this course can be categorized according to two distinctions. Some models consider continuously varying quantities, such as temperatures and forces, and others consider discrete quantities, such as populations or packets of information. Some models are intended to describe dynamic systems, such as springs bouncing or economies growing, and others describe static systems, such as a balancing of forces. Different combinations of characteristics will typically involve applying material from different areas of mathematics. For example, the study of continuous dynamic systems often involves eigenvalue analysis of differential equations, whereas the study of continuous static systems may focus on minimizing some kind of energy functional. Discrete dynamical systems are sometimes modeled using iterated matrix multiplications and linear algebra techniques, whereas discrete static systems typically involve combinatorial graph theory.

Different instructors will choose to highlight different phenomena to model. Some commonly chosen topics include: spring systems, electrical circuits, population growth for single species, population growth for multiple species, and network flow. Mathematical techniques to be covered will typically include some or all of the following, and possibly others as well: dimensional analysis, minimization of linear and quadratic functionals in multiple variables, differential equations, eigenvalue analysis and decoupling of equations, analysis of periodic and other long-term behavior, difference equations, chaos, combinatorial graph theory, max-flow/min-cut theorem and matchings in combinatorial graphs.

Prerequisites: Math 3342 (and its prerequisite courses) and one of CSCI 1011, 1041, 1111, 1121 or 1131.
MATH 3410, MATHEMATICS OF FINANCE

(This description is being written.)

Prerequisites: Math 2233.
MATH 3411, STOCHASTIC CALCULUS METHODS IN FINANCE

(This description is being written.)

Prerequisites: Math 2184 or 2185; Math 3410.
Numerical analysis is the study of algorithms that use numerical approximations (as opposed to general symbolic manipulations) for the problems of mathematical analysis. The overall goal of the field of numerical analysis is the design and analysis of techniques to give approximate but accurate solutions to hard problems. The field of numerical analysis includes many sub-disciplines. In Math 3553, we discuss some of the major ones.

- **Accuracy and precision**: The arithmetic performed by a calculator or computer is different from that in algebra and calculus courses. In our traditional mathematical world, we permit numbers with an infinite number of digits. In the computational world, however, each representable number has only a fixed, finite number of digits. The error that is produced when a calculator or computer is used to perform real number calculations is called round-off error. An algorithm is considered good when it keeps accuracy under computer arithmetic. We discuss some fundamental concepts such as round-off errors, computer arithmetic, convergence of algorithms etc.

- **Interpolation**: This is a method of constructing new data points within the range of a discrete set of known data points. In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate (i.e., estimate) the value of that function for an intermediate value of the independent variable. We will introduce the polynomial interpolation which can be represented in Lagrange or Newton form. We study the error estimate for such interpolation.

- **Solving nonlinear equations**: For a general function $f(x)$, one usually cannot solve $f(x) = 0$ to find zeros analytically. On the other hand, we can use numerical methods to approximately solve for it. In this course, we will study bisection method and Newton method (fixed-point iteration method). The error analysis and the rate of convergence for such methods are discussed.

- **Numerical integration**: This constitutes a broad family of algorithms for calculating the numerical value of a definite integral. We will study the Newton-Cotes quadrature for uniform quadrature nodes. Then adaptive quadrature methods such as Gaussian quadrature are explored.

- **Direct Methods for linear systems**: The problem $Ax = b$ arises in many areas of science and engineering. We introduce the Gauss elimination method, which induces LU decomposition, and Gauss elimination method with pivoting. For some matrices with special structures such as symmetric positive definite matrices, we study the Cholesky factorization method.

- **Iterative methods for linear systems**: These are usually more efficient than the direct methods. We discuss the Jacobi and Gauss-Siedel iterations and the SOR method. Conjugate gradient method are explored as well.

- **Approximation of functions**: These are discussed by exploring the least squares approximation, orthogonal functions such as Legendre polynomials, Chebyshev polynomials and Laguerre polynomials. We briefly discuss the rational function approximation.

Prerequisites: Math 2184 or 2185, and Math 2233, and one of CSCI 1011, 1041, 1111, 1121 or 1131.
Very roughly, combinatorics deals with finite objects and their structure. However, combinatorics is such a broad field that this description leaves much out. The main topic in Math 3613 is easier to nail down: the course focuses on questions of the type “How many ... are there?”, as we illustrate below.

(a) How many ways are there to cover all squares in an $n \times m$ grid with dominoes, where each domino covers two squares and no two dominoes cover the same square?
(b) How many partitions of an $n$-element set into $k$ subsets are there?
(c) How many ways are there to dissect a regular $(n+2)$-gon into triangles?
(d) How many ways can we write a positive integer $n$ as a sum of positive integers, written in non-increasing order? There are five for $n = 4$: $4$, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$.

When we consider sequences of such problems, we often see structure that may not be apparent in an isolated instance. For instance, in problem (a) fixing $n$ to be 2 and letting $m$ vary, with $m \geq 0$, yields the sequence of answers starting with 1, 1, 2, 3, 5, 8, 13. If $f_m$ denotes the general term, then $f_m = f_{m-1} + f_{m-2}$. Why? This recurrence relation reflects structure. In problem (b), let $S(n,k)$ denote the number of partitions. If we fix $n$ and let $k$ vary from 0 to $n$, we get the polynomial identity

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)(x-2)\cdots(x-k+1).$$

Why? Again, this algebraic relation reflects structure. This example hints at the fact that a key aspect of combinatorics is a powerful interplay with algebra. If $C_n$ denotes the solution to problem (c) in case $n$, then we get the relation $C_n = C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0$. Why? This recurrence relation reflects structure, and it is useful since it leads to an infinite series displaying $C_n$:

$$\frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 0} C_n x^n,$$

from which we can find $C_n$. The number $p(n)$ that is the answer to problem (d) is easily expressed using a series and an infinite product:

$$\sum_{n \geq 0} p(n)x^n = \prod_{k \geq 1} \frac{1}{1 - x^k} = \frac{1}{1 - x} \cdot \frac{1}{1 - x^2} \cdot \frac{1}{1 - x^3} \cdot \frac{1}{1 - x^4} \cdots$$

Why? Again, algebra reflects important structure. We can learn much about $p(n)$ through such representations. With the insight gained by studying combinatorics, we can look at identities such as

$$\prod_{i \geq 0} (1 + x^{2i+1}) = \sum_{d \geq 0} \left( \prod_{i=1}^{d} \frac{1}{1 - x^{2i}} \right) x^d$$

and get a conceptual understanding of them by seeing what each side counts.

Besides being inherently intriguing, the questions that combinatorics treats and the tools that it develops have applications in many fields, including probability, statistics, and computer science. Indeed, the historical roots of combinatorics include finding probabilities in games of chance (e.g., what is the probability of getting certain hand in a game of cards?).


For a fuller, yet still brief, description, see https://en.wikipedia.org/wiki/Combinatorics

Prerequisites: Math 2971.
Math 3632, Introduction to Graph Theory

It is often useful to depict data with diagrams made of dots and lines. For example, think of a family tree, an evolutionary tree, a diagram of administrative structure, a social network, the maps in airline magazines showing routes, or a network of roads. These are all examples of graphs, which we can represent as diagrams of vertices (dots) and edges. Applications are ubiquitous. For instance, to get a computer to play a game well, it is programmed to generate and analyze a graph in which the vertices are the possible states in the game and edges indicate which state can be reached from another in one move; the computer can then search this graph to find its optimal move.

Math 3632 focuses on theory, but the range of applications is enormous, and they motivate many of the topics considered.

If the graph on the left above represents committees, with an edge between vertices indicating that the committees share a member, how few time slots do we need in order to schedule meetings so that no person has a conflict? This motivates the graph coloring problem: what is the fewest number of colors needed so that we can assign colors to the vertices so that no vertices joined by an edge have the same color? Associated to a graph is a polynomial, its chromatic polynomial, whose value at each positive integer \( k \) gives you the number of valid colorings of the graph with a set of \( k \) colors.

If the graph on the right represents which applicants (the \( y \)s) are qualified for which jobs (the \( x \)s), what is the maximum number of jobs we can successfully fill? What is the maximum number of applicants who can get jobs? These are question in matching theory. (The red edges in the illustration give a matching that satisfies all applicants.) Such questions are related to problems about selecting minimal sets of vertices for which every edge has an end-vertex in the set.

An electrical circuit is another example of a graph. If we want to print circuits on chips, it would be useful to know how few layers we need in order to print the circuit. In particular, can the circuit be drawn in the plane with no crossings (except where intended, at vertices)? This is the issue of planarity. For instance, the graph on the left above is not drawn in a planar manner, but it has a planar drawing: move the third vertex in the middle row below the bottom row, and replace the edge to its upper neighbor by an edge that curves around the left side of the graph. It turns out that there are two special graphs that completely control which graphs have planar drawings.

For a graph that depicts streets, you might ask for the number of different walks from your home to your favorite restaurant, no two of which use the same street. For a communications network, you might ask how many nodes can fail before messages can no longer be delivered among the remaining nodes? These are issues of graph connectivity. Results about connectivity are the work-horses behind many of the deepest results in graph theory.

Graph theory has a wealth of open problems. For instance, it is known that each planar graph can be drawn with edges that are straight lines, but it is still a conjecture that there is such a drawing in which the length of each edge is an integer (Harborth’s conjecture). See

http://www.openproblemgarden.org/category/graph_theory and

For a fuller, yet still brief, description, see https://en.wikipedia.org/wiki/Graph_theory

Prerequisites: Math 2971.
Prerequisites: Math 2971.
MATH 3720: AXIOMATIC SET THEORY

(This description is being written.)

Prerequisites: Math 2971.
MATH 3730: COMPUTABILITY THEORY

(This description is being written.)

Prerequisites: Math 2971.
MATHEMATICS 3740: COMPUTATIONAL COMPLEXITY

(This description is being written.)

Prerequisites: Math 2971.
Differential geometry explores the intrinsic geometry of curves and surfaces using methods from differential and integral calculus, and linear algebra. We begin with the study of planar curves and space curves parameterized by arc length, and continue onto the study of regular surfaces. This requires some knowledge of continuity and differentiability of functions and mappings in two and three-dimensional space. Roughly speaking, a regular surface is obtained by taking pieces of a plane and deforming them in such a way that there are no sharp points, rough edges, or self-intersections. The idea is to define a two-dimensional structure $S$ where the tangent plane at $p \in S$ is well-defined and we can extend the usual notions of differential calculus to functions and vector fields on $S$.

The rate of change of the direction of the tangent line to a curve $C$ leads to the definition of the curvature of $C$. How quickly a surface pulls away from its tangent plane is measured using the unit normal $N(p)$, also called the Gauss map. The differential of the Gauss map at $p$ (denoted $dN_p$) is a linear self-adjoint map; its invariants are the Gauss curvature $K$ and the mean curvature $H$. A surprising result is that $K$ can be computed using only quantities related to the metric on $S$ and its first derivatives. This result is closely related to Gauss’s Theorema Egregium: $K$ is invariant by local isometries. This means that if one bends a surface without stretching, the Gauss curvature $K$ does not change. This has consequences for the cartographer in that any (flat) map of the earth must distort distances!

The generalization of a “straight line” to curved surfaces is the geodesic. For a regular surface, geodesics are (locally) the shortest path between points in $S$. You might be surprised by what is the shortest path from Dulles (IAD) to Seoul (ICN) (check out an app such as DistanceCalculator). Another surprise is that for a surface with curvature $K$, the interior angles $\{\phi_1, \phi_2, \phi_3\}$ satisfy

$$\begin{align*}
\phi_1 + \phi_2 + \phi_3 &> \pi, \quad \text{when } K > 0 \text{ (space positively curved)}, \\
\phi_1 + \phi_2 + \phi_3 &= \pi, \quad \text{when } K = 0 \text{ (flat space)}, \\
\phi_1 + \phi_2 + \phi_3 &< \pi, \quad \text{when } K < 0 \text{ (space negative curved)}. 
\end{align*}$$

This is related to one of the deepest theorems in differential geometry, the Gauss-Bonnet Theorem. A corollary of the Gauss-Bonnet Theorem states that for an orientable compact surface $S$ with surface measure $d\sigma$,

$$\int_S K d\sigma = 2\pi \chi(S)$$

where $\chi(S) = 2 - 2g$ is the Euler-Poincaré characteristic of $S$ and $g$ is the number of handles of $S$.

The subject of differential geometry provides an excellent path to transition to higher mathematics in the study of ordinary and partial differential equations, the calculus of variations, complex analysis, topology, Riemannian geometry, and differentiable manifolds (a generalization of the notion of a regular surface). The list of applications of differential geometry continues to grow, ranging from problems in machinery design, to the classification of four-manifolds, to the fundamental forces in nature, and to the study of DNA.

Prerequisites: Math 2184 or 2185; Math 2233 and Math 2971.
In your mathematics education, you first learned about numbers and operations on them, and their properties (for example, addition is associative, as is multiplication, and multiplication distributes over additions). Later you learned about polynomials and operations on them, and you saw that many of the same properties hold, including the three just cited. Abstract algebra is the general study of such systems. In abstract algebra, we study sets and operations on those sets that are subject to certain rules, such as those cited above. We work in an abstract setting; we illustrate the theory with, and apply it to, particular examples, but our interest is in proving results that hold in all examples. The proofs we give in the general setting apply to all examples, which makes for great efficiency. It also gives us insight into what holds in general versus what depends on the details of a given example.

Rings are a class of objects studied in abstract algebra that include many familiar examples. Rings have operations of addition and multiplication defined on a set, subject to familiar rules, except that we do not require non-zero elements to have multiplicative inverses. An example you know is the set \( \mathbb{Z} \) of integers with the usual operations; \( \frac{1}{2} \) has no multiplicative inverse since \( \frac{1}{2} \not\in \mathbb{Z} \). In general rings, multiplication might not commute, so we can also consider rings of \( n \) by \( n \) matrices (matrix multiplication in general does not commute).

We also study less constrained structures, specifically, structures with just one operation. In the structure called a group, we have a set, one operation, and that operation is associative, has an identity, and each element has an inverse. Symmetry groups are one of the many important examples of groups. For instance, in a geometric figure, such as a regular pentagon or a tetrahedron, we can take two symmetries and follow one by the other (as in function composition). For instance, we can rotate the tetrahedron by 120° about the line that passes through a vertex and the center of the opposite face, and follow that by a 180° rotation about the line that passes through the centers of opposite sides. Each such symmetry has an inverse (in the first case, rotate another 240° about the same line; in the second case, apply the same rotation again). Another familiar example of a group is the collection of subsets of a fixed set with the operation of symmetric difference: \( A \triangle B = (A \cup B) - (A \cap B) \). In that example, \( \emptyset \) is the identity and each set is its own inverse.

Math 4122 often treats two very striking examples of using algebra to solve classical problems: the impossibility of trisecting general angles with a ruler and compass, and proving that there cannot be counterparts of the quadratic equation for polynomials of degrees five or higher.

Abstract algebra is used in many other fields. It has very strong connections with number theory. In topology, one associates groups with topological spaces with the aim of distinguishing between the spaces by distinguishing between the associated groups. Abstract algebra has many important applications to a wide variety of other areas, including cryptography, coding theory, and physics.

Abstract algebra has many classical and new open problems. Much research now focuses on the interface between algebra and other fields. See

http://www.openproblemgarden.org/category/algebra
http://www.openproblemgarden.org/category/group_theory and

For a fuller, yet still brief, description, see https://en.wikipedia.org/wiki/Abstract_algebra

Prerequisites for Math 4121: Math 2971 and either Math 2184 or 2185.
Prerequisites for Math 4122: Math 4121.
The word analysis in mathematics refers to the study of functions and their limits. Hence real analysis is the study of functions of a real variable or of several real variables. The first college-level course in analysis is calculus 1 and 2, the familiar first-year course about differentiation and integration of functions of a single real variable.

Math 4239, the first semester of the real analysis sequence, revisits the same ideas as are discussed in calculus 1 and 2: limits, continuity, differentiation, integration, sequences, and series. The course is sometimes called Advanced Calculus. (In this context, we refer to the calculus 1 and 2 sequence as Elementary Calculus.) Even though the topics in advanced calculus seem to be the same as in elementary calculus, the perspective is so different that it sometimes seems to students that they are entirely different subjects. In the elementary courses, almost every problem is about some formula, like \( x \sin x \) or \( e^{-x^2} \) or \( \arctan(\ln(x + 1)) \), and the goal is to execute some computation with that formula. In Math 4239, hardly any problems ask specifically about one formula; instead, you learn what must be true about any formula. So in the elementary calculus sequence, a typical problem might begin “Suppose \( f(x) = x^2 + \tan(x) \).” In Math 4239, a typical problem might begin “Suppose \( f \) is a continuous function from \( \mathbb{R} \) to \( \mathbb{R} \).” This is the abstract viewpoint, which we adopt because it is efficient and enlightening.

Math 4239 investigates the foundations behind what is taught in calculus 1 and 2. We investigate various conditions that ensure the existence of limits, derivatives, and integrals. In calculus 1 and 2, we start with nice formulas that tend to automatically guarantee the existence of such things, and we simply compute them. The focus of Math 4239 is the assortment of theorems that assert such guarantees. So real analysis is the theory behind calculus.

Here are a few questions that illustrate ideas found in Math 4239:

- Can a function from \( \mathbb{R} \) to \( \mathbb{R} \) be continuous at all \( x \) but differentiable at no \( x \)?
- Can a function from \( \mathbb{R} \) to \( \mathbb{R} \) be continuous at every irrational number but continuous at every rational number?
- Does every continuous function on \([0, 1]\) have a definite integral?
- Under what circumstances does a Maclaurin series represent the function from which it came?

To analyze such questions, certain concepts are introduced that typically are not seen in the elementary calculus sequence. These concepts include open and closed sets, completeness, compactness, liminf and limsup, bounded variation, uniform continuity, and uniform convergence.

Part II of the course, Math 4240, covers the theory behind multivariable calculus. We develop an understanding of the derivative as a linear operator, with the familiar chain rule manifesting itself as a theorem about the product of matrices. We rework the definition of the Riemann integral in more than one dimension. We state, prove, and use the implicit and inverse function theorems. And if time permits, we may introduce the idea of a differential form in order to understand what is called the generalized Stokes theorem, which is a grand generalization and unification of the fundamental theorem of calculus, the fundamental theorem of line integration, Green’s theorem, Gauss’s theorem, and Stokes’s theorem.

The real analysis courses sit centrally between pure and applied mathematics. They display the abstraction and rigor of pure mathematics, but the objects of study are at the center of all applied mathematics. Every student contemplating post-graduate study in mathematics (pure or applied), physics, or economics should complete this sequence, or at the very least the first half of it.

Prerequisites for Math 4239: Math 1232 and Math 2971.
Prerequisites for Math 4240: Math 2184, Math 2233, and Math 4239.