

## Brief Sketches of Post-Calculus Courses

The bulletin descriptions of courses often give you little idea about the topics of the courses, unless you happen to already know the many technical terms that the descriptions use. To address this gap, we offer brief, largely self-contained sketches that are written to give you a glimpse of our upper-level offerings. It may be especially useful to read these sketches around registration time, when you are deciding which courses to take next semester, or when planning for future courses.

At the start of this document, we offer some possible routes for starting the mathematics major. While that may help you get started, we encourage all students who are considering the mathematics major to talk with an advisor in the department as early as possible so that you can get advice that is tailored to your background and to your academic and career goals.

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## SUGGESTIONS FOR GETTING STARTED ON THE MATHEMATICS MAJOR

A first and very important piece of advice is to talk with an advisor in the Mathematics Department early on — it is never too early to consult the advisors, and we like to talk with students. We offer general advice below, but we stress that it is best to consult an advisor so that your academic and career goals, your ambitions, and your background can be taken into account. To meet with an advisor in the Mathematics Department, send an e-mail to [mathadv@gwu.edu](mailto:mathadv@gwu.edu) and let us know what times you are free to meet.

For all students, when possible, we recommend taking UW 1020 (University Writing) in your Freshman Fall so you can get WID credit if you decide to take Math 2971W in your Freshman Spring. (See GW's [WID Policies](#), especially items 2 and 3 on that web page.)

Students start with different backgrounds, so we outline three options to fit the situations that we see most often. For help with other situations (e.g., later starts on the major), please consult an advisor. All suggestions below meet the **goal of completing all prerequisite courses** (the 1000- and 2000-level courses) **by the end of your sophomore year**; ideally, you will also then have some start on the upper-level courses.

For students who need or want to start with Math 1231, Calculus 1:

- Freshman Fall: Math 1231,
- Freshman Spring: Math 1232 and CSCI (see note (b) below),
- Sophomore Fall: Math 2233 and Math 2971W,
- Sophomore Spring: Math 2185 and a 3000-level Math course.

See notes (b)–(d) below.

For students who have AP credit for Math 1231 and want to start with Math 1232:

- Freshman Fall: Math 1232,
- Freshman Spring: Math 2233 and Math 2971W,
- Sophomore Fall: Math 2185 and a 3000-level Math course,
- Sophomore Spring: CSCI (see note (b) below) and a 3000-level Math course.

See notes (a)–(d) below.

For students who have AP credit for Math 1231 and 1232, and want to start with Math 2233:

- Freshman Fall: Math 2233,
- Freshman Spring: Math 2971W,
- Sophomore Fall: Math 2185 and a 3000-level Math course,
- Sophomore Spring: CSCI (see note (b) below) and a 3000-level Math course.

See notes (a)–(d) below.

### Notes:

- (a) If you believe that you have the equivalent of Math 1231, or of Math 1231 and 1232, but lack the AP credit (e.g., via IB courses), please discuss your situation with an advisor.
- (b) The Computer Science requirement consists of taking one of CSCI 1011 (Java), CSCI 1012 (Python), CSCI 1041 (FORTRAN), CSCI 1111 (Software Development), CSCI 1121 or 1131 (C). You can delay this if you do not plan to take Math 3553 in your Junior Fall. In the Pure Concentration, this can be replaced by an additional Math elective.
- (c) If you have taken or are considering taking Math 2184, please consult an advisor. Math 2185 is required for the BS degree and is preferred for the BA; Math 2184 is accepted for the BA.
- (d) While we suggest taking some 3000-level Math courses in your sophomore year, this is not required.

## A COMPARISON OF THE COMPUTER PROGRAMMING OPTIONS

The two main options to satisfy the programming requirement are CSCI 1011, *Introduction to Programming with Java*, and CSCI 1012, *Introduction to Programming with Python*. Both Python and Java are among the most widely-used programming languages today, both are very powerful and versatile, and both are expected to be main programming languages well into the future. Below we make brief comments that are intended to help you decide between the two.

We recommend Python for most students. It is easier for programming novices to learn.

Python is *interpreted* while Java is *compiled*. Roughly, this means that, as it runs, your Python program is translated into a format that the computer understands; for Java, you first run a separate translation process before you run your program. This has several implications. Partly because of the run-time translation, Python may sometimes be somewhat slower (this is not an issue for most applications, especially given how fast computers are). One advantage of interpreted languages, such as Python, is that it is easy to test your code by giving the computer small parts of your program in an interactive mode and seeing whether it produces what you expect. One can also use interpreted languages such as Python as sophisticated calculators to run small tasks that one wouldn't write a whole program for.

Since Java is compiled, data types are static, that is, you must specify what type of data each variable can hold — an integer, a floating-point number (a number with a decimal point), or a string (a sequence of characters, such as a word or sentence). In Java, putting a different type of data into a variable causes an error. In contrast, Python is dynamically typed, so the program figures out what data type to use based on the data you give it. This makes Python programs easier to write and less cluttered; the drawback is that if you do not observe the relevant conventions, you can get (possibly confusing) errors.

Invariably, complex programs, or even simple programs written by beginners (or even experienced programmers), contain errors that must be tracked down and fixed — this is the process of debugging (see the origin of the term at [debugging](#)). Both Python and Java have extensive support for debugging. For instance, special modes allow you to step through a Python program one command at a time to see the results. There is also extensive support for both languages on the web, such as [PythonTutor.com](#).

Both Python and Java have enormous [libraries](#) of mathematical functions that you can import into your programs, so you do not have to create everything from scratch. For instance, the Python [math](#) library gives you access to the functions that you know (e.g., trigonometric functions, logarithms, and factorials), [cmath](#) gives extensive support for complex numbers, and [itertools](#) allows you to generate subsets, permutations, and combinations easily. The routines in these libraries are optimized to run very fast.

Python is the language behind the SageMath project, which is an open-source counterpart of programs like Mathematica, Maple, and Matlab. If you know Python, Sage will come quickly.

Through either language, you will learn all of the basic programming concepts and so will be in a good position to learn other computer languages, should you develop the need or interest to do so.

For more detailed comparisons and some samples of code, click on these links:

[Ben Putano, Java vs. Python: Coding Battle Royale](#),  
[Steve Ferg, Python & Java: A Side-by-Side Comparison](#), and  
[John Mueller, Java vs. Python: Which Programming Language Is Best?](#)

For a more mathematically-oriented introduction to Python than most books, see <http://hplgit.github.io/scipro-primer/>.

The other programming courses offered by the Computer Science Department — CSCI 1041 (FORTRAN), CSCI 1111 (Software Development), CSCI 1121 or 1131 (C) — are sometimes restricted to SEAS students. FORTRAN is very old (but still fairly widely used). C is noted for its speed, but also for its steep learning curve.

MATH 2185, COMPREHENSIVE INTRODUCTION TO LINEAR ALGEBRA

Consider the following objects and operations.

- For polynomials with coefficients in the real numbers,  $\mathbb{R}$ , you know how to add two polynomials, and also how to multiply a polynomial by a real number. Both operations yield another polynomial. You can do the same with polynomials over the rational numbers,  $\mathbb{Q}$ , if you multiply them only by rational numbers. You can also replace  $\mathbb{R}$  by the complex numbers,  $\mathbb{C}$ .
- For functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  (that is, the domain and range is  $\mathbb{R}$ ), you know how to add two functions, and to multiply a function by a real number, and the result is another real-valued function of a real variable. As above, you can replace  $\mathbb{R}$  by  $\mathbb{Q}$  or  $\mathbb{C}$ . Also, for  $\mathbb{R}$ , you can focus just on the functions that are differentiable everywhere, or those that are integrable on closed intervals.
- You might know about adding two elements in  $\mathbb{R}^3$ , and multiplying by a real numbers, using the operations  $(a, b, c) + (r, s, t) = (a + r, b + s, c + t)$  and  $k(a, b, c) = (ka, kb, kc)$ .

Certain properties are common to all such examples: for instance, addition is commutative and associative. The same structure (a set of objects with operations of addition of these objects and multiplication of them by numbers, all obeying certain rules) appears in many important contexts in addition to those above. The first focus of attention in linear algebra, *vector spaces*, abstracts these examples. In a vector space, we have a set (whose elements we call vectors) and a field (such as  $\mathbb{R}$  or  $\mathbb{Q}$  or  $\mathbb{C}$ ) and operations of addition of two vectors and multiplication of a vector by a *scalar* in the field (e.g., by a number in  $\mathbb{R}$ ), satisfying eight familiar rules.

Using the operations of addition of polynomials and multiplication by numbers, every polynomial can be written in exactly one way in terms of the polynomials in the set  $\{1, x, x^2, \dots\}$ . Likewise, each vector in  $\mathbb{R}^3$  can be written in exactly one way as a sum of scalar multiples of the vectors in the set  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ ; in particular,  $(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$ . These are examples of *bases* of vector spaces. Each vector in a vector space can be described uniquely by its coordinates relative to a given basis. A vector space has many different bases, but, in a given vector space, all bases have the same number of vectors; that number is the *dimension* of the vector space. Bases, dimension, and related notions occupy much of the first part of Math 2185.

Two formulas that you know from calculus,

$$\frac{d}{dx}(f(x) + k \cdot g(x)) = f'(x) + k \cdot g'(x) \quad \text{and} \quad \int (f(x) + k \cdot g(x)) dx = \int f(x) dx + k \int g(x) dx,$$

show that some very important operations preserve sums and scalar multiples. Generalizing this, the second focus of attention in linear algebra is functions  $T$  from one vector space to another, over the same field, that preserve the operations in the sense that  $T(u + av) = T(u) + aT(v)$  for all vectors  $u$  and  $v$  in the domain, and elements  $a$  of the field. Such functions are called *linear transformations*.

Matrices make many appearances in linear algebra, starting with the basic problem of solving systems of linear equations (which can be cast as matrix equations). For a linear transformation  $T : V \rightarrow U$  between finite-dimensional vector spaces, matrices provide efficient ways to obtain the coordinates of the image  $T(v)$  in a given basis of  $U$  from the coordinates of the vector  $v$  in a given basis of  $V$ . The connections between linear transformations and matrices leads to a rich theory of decomposing linear transformations  $T : V \rightarrow V$  into simpler pieces.

This course and Math 2184 have much in common (so credit may not be earned for both), but the perspective is different, with Math 2184 emphasizing calculation, and Math 2185 emphasizing theory, proof, and an abstract point of view. Among the things we gain with the more general setting of Math 2185 are (a) efficiency: one proof in the general setting replaces separate proofs in each of the particular examples, and (b) insight: we find out which properties of interest in a given example hold in general and which are particular to that example. The abstract point of view offered in Math 2185 also prepares students to understand important applications of linear algebra in several other fields (e.g., computer science, physics, and economics), some of which are covered in Math 3125.

Prerequisites: Math 1231 and Math 2971 (Math 2971 and 2185 may be taken simultaneously).

## MATH 2971, INTRODUCTION TO MATHEMATICAL REASONING

The great advantages of organizing our mathematical knowledge into chains of deductions, going from definitions and explicitly-stated assumptions (axioms) to deep and powerful conclusions, have been widely recognized for more than two and a half millennia. The resulting structure solidifies our knowledge and makes it easier to digest, appreciate, and remember what we know, to share mathematics with each new generation, and to observe patterns that suggest directions for further development. Mathematical proofs serve as the mortar that makes this structure rock-solid, and they provide the explanations of why everything works. Proofs and proof-related skills are the focus of Math 2971.

In many ways, Math 2971 is a gateway — a passage to new horizons.

- It gives an introduction to how to read, understand, devise, and write mathematical proofs. In particular, it carefully analyzes and illustrates the process of coming up with the ideas that go into constructing proofs.
- Besides treating basic logic and proof techniques (e.g., quantifiers, proof by contradiction, induction, using the contrapositive), it covers many fundamental structures and concepts that you will be assumed to be familiar with in upper-level courses (e.g., basic set theory, equivalence relations, set partitions, cardinality). Once you have mastered these topics in Math 2971, you can focus on the really new ideas in upper-level courses.
- This course expands your understanding of what mathematics is. While there is not sufficient time to include surveys of numerous parts of math, the course uses examples drawn from a variety of branches of math (with enough background provided to make the examples accessible) and so gives you a glimpse of some of the fascinating worlds that lie beyond the introductory-level courses.
- Math 2971 fosters skills to tackle non-routine problems.
- As a WID course, it provides you with practice and feedback that promote clear and precise writing in the style used by professional mathematicians.

For many reasons, it is desirable and highly recommended to take Math 2971 as early as possible, either at the same time as Math 1232 (Calculus II) or just after that course. In addition to being a crucial prerequisite for many of our upper-level courses, this course exposes you to what lies beyond the subjects that you are already familiar with and gives you a different perspective on math; as a result, it can help you decide which of our three tracks for the mathematics major (pure, applied, and interdisciplinary) may work best for you.

The target audience for this course is declared or prospective mathematics majors and minors. Everyone who enrolls in this course should be eager to grapple with challenging mathematical ideas and problems. This is a WID course, so expect a fair bit of writing. Be aware of GW's [WID Policies](#), especially items 2 and 3 on that web page. (Those who are not considering mathematics as a major or minor should talk with the instructor before enrolling in this course since those looking just to fulfill a WID requirement could find this course too deep.)

Prerequisite or co-requisite: Math 1232.

Recommended follow-up courses for practicing your proof-writing and problem-solving skills: Math 2185 or any 3000-level, theoretically-oriented class.

Math 2185, Comprehensive Intro. to Linear Algebra	Math 3120, Elementary Number Theory
Math 3125, Linear Algebra II	Math 3257, Complex Variables
Math 3613, Combinatorics	Math 3632, Graph Theory
Math 3710, Mathematical Logic	Math 3720, Axiomatic Set Theory
Math 3730, Computability Theory	Math 3740, Computational Complexity
Math 3806, Topology	Math 3848, Differential Geometry

(Many students benefit from taking one or more of these courses before taking 4000-level courses.)

## MATH 3120, ELEMENTARY NUMBER THEORY

Number theory treats properties of the positive integers. “Elementary” in elementary number theory reflects the use of relatively basic techniques (not the level of difficulty), in contrast to, for instance, algebraic and analytic number theory, which use higher-level algebra or analysis.

You might already have seen some beginning topics from Math 3120 in Math 2971, such as the division algorithm (a basic result that is behind many topics in number theory), the fundamental theorem of arithmetic (every positive integer factors uniquely into primes), the equivalence relation of congruence modulo an integer  $n$ , and the fact that there are infinitely many primes. Below we describe some other topics that elementary number theory treats.

A strengthening of the statement that there are infinitely many primes is that the series

$$\sum_{\text{primes } p} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$$

diverges. This series is a variation on the harmonic series, but thinned out to just the reciprocals of the primes. The divergence is proven in elementary number theory. Such results give us some idea of the relative density of primes among the integers. For instance, it shows that primes are denser than squares since  $\sum_{n \geq 1} \frac{1}{n^2}$  converges.

Primes have a wealth of interesting properties. To cite just two examples, a prime  $p$  always divides  $1 \cdot 2 \cdot 3 \cdots (p-1) + 1$ , that is,  $p | (p-1)! + 1$ . For instance,  $4! + 1 = 25$ , which is divisible by 5. Also, for any prime  $p$  and any integer  $a$  that is not a multiple of  $p$ , the number  $a^{p-1} - 1$  is always divisible by  $p$ . For instance,  $7 | 53^6 - 1$ . Why do such properties always hold? Such properties and their consequences are explored in elementary number theory.

The second property of primes just cited has a counterpart that replaces the prime  $p$  by any positive integer  $n$ , and the exponent  $p-1$  by  $\phi(n)$ , which is the number of integers not exceeding  $n$  that have no prime factors in common with  $n$ ; the integer  $n$  divides  $a^{\phi(n)} - 1$  when  $a$  and  $n$  have no common factors. These ideas play key roles in one of the major applications of number theory to cryptography. Number theory is the basis of the currently best methods for encrypting data, such as your e-mail.

In elementary number theory, we show how to generate all Pythagorean triples, that is, all triples  $a, b, c$  of integers (such as 3, 4, 5, or 5, 12, 13) that satisfy the Pythagorean relation  $a^2 + b^2 = c^2$ . This brings up the topic of writing numbers as sums of squares. Notice that the primes 3, 7, and 11 cannot be written as the sums of two squares, while

$$2 = 1^2 + 1^2, \quad 5 = 2^2 + 1^2, \quad 13 = 3^2 + 2^2, \quad 17 = 4^2 + 1^2.$$

Which primes are sums of two squares? Elementary number theory yields the answer, and from that, we deduce which positive integers  $n$  are the sums of two squares.

Observe that we can write 6 as the sum of three squares,  $6 = 2^2 + 1^2 + 1^2$ , while 7 requires four squares,  $7 = 2^2 + 1^2 + 1^2 + 1^2$ . Do some integers require more squares? No! In elementary number theory, we show that one never needs more than four squares: for any positive integer  $n$ , the equation  $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$  has an integer solution.

Number theory has a wealth of tantalizing open problems, ranging from classical problems (e.g., the twin prime conjecture (there are infinitely many pairs of primes of the form  $p, p+2$ , such as 3 and 5, or 5 and 7) and Goldbach’s conjecture (every even integer 4 and greater is the sum of two primes; e.g.,  $24 = 13 + 11$ )) to recent problems. See

[http://www.openproblemgarden.org/category/number\\_theory\\_0](http://www.openproblemgarden.org/category/number_theory_0) and

[https://en.wikipedia.org/wiki/List\\_of\\_unsolved\\_problems\\_in\\_mathematics](https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics).

For a fuller, yet still brief, description, see [https://en.wikipedia.org/wiki/Number\\_theory](https://en.wikipedia.org/wiki/Number_theory).

Prerequisites: Math 2971.

## MATH 3125, LINEAR ALGEBRA II

Like its prerequisite Math 2185, this course is a theory-oriented study of linear algebra that deepens students' understanding of vector spaces and linear transformation in particular, and rigorous mathematics in general. This course also introduces several fundamental applications of linear algebra to other fields. The choice of topics is somewhat flexible, but the following describes some key areas that are typically covered.

You know that if  $V$  is a vector space of dimension  $m$  and  $T: V \rightarrow V$  is a linear transformation, then choosing a basis for  $V$  allows us to write down an  $m \times m$  matrix  $A_T$  that captures the action of  $T$ . But what *is* that action? Is the effect of  $T$  to rotate, flip, stretch, shrink, some combination of these, or something else? A large portion of the material in Math 3125 can be seen as providing ways to answer this question. For instance, if  $W$  is a subspace of  $V$  such that every vector in  $W$  is mapped by  $T$  to a vector in  $W$  (possibly the same vector, but not necessarily), then we call  $W$  a  *$T$ -invariant subspace*. The simplest example of this is an eigenspace; if  $T(v) = \lambda v$  for some scalar  $v$ , then the linear span of  $v$  is a 1-dimensional  $T$ -invariant subspace. Understanding the relationships between the  $T$ -invariant subspaces allows us to choose a basis for  $V$  such that the structure of the matrix  $A_T$  reveals important information about the action of  $T$  on  $V$ . For example, if  $V$  has a basis consisting of eigenvectors of  $T$ , then  $A_T$  will be diagonal. Jordan normal form, a topic in Math 3125, provides an approach in terms of invariant subspaces for many cases when  $V$  has no basis of only eigenvectors.

Another important concept that highlights linear transformations enjoying a particular type of action is that of inner product and, more generally, bilinear form. Let  $\mathbb{F}$  denote the field of scalars for  $V$ . A function  $\varphi: V \times V \rightarrow \mathbb{F}$  is called *bilinear* if it is linear in each coordinate; for vectors  $v, w$  in  $V$  we can view  $\varphi(v, w)$  as providing some sort of measurement of the relationship between  $v$  and  $w$ . With some additional properties,  $\varphi$  will be an inner product, and thus will provide us with a notion of distance and angle in  $V$ . Given this we can ask what effect a linear transformation  $T: V \rightarrow V$  will have on the lengths of vectors and the angle between them. Linear transformations that leave lengths and angles unchanged are called *orthogonal* when  $\mathbb{F} = \mathbb{R}$  (real numbers) and *unitary* when  $\mathbb{F} = \mathbb{C}$  (complex numbers). Not surprisingly, orthogonality and unitarity tie in with many special properties. In Math 3125 you will learn about this, and in particular you will learn how orthogonal and unitary transformations can be decomposed in revealing ways.

In Math 2185 you were exposed to the idea that linear transformations can be viewed as elements of a vector space. Of course, this means that the reverse is true as well! Given a vector  $v$  in a vector space  $V$  over the scalar-field  $\mathbb{F}$  we can use  $v$  to define a function  $V \rightarrow \mathbb{F}$  by the rule  $w \mapsto v^T w$  for all  $w \in V$ . More generally, we can define a vector space  $V^*$  called the *dual* of  $V$  consisting of all linear functions  $V \rightarrow \mathbb{F}$ . When  $V$  is finite dimensional, all elements of  $V^*$  arise in the way described above, but the story is more complicated for infinite dimensional vector spaces. The vector space dual is a useful tool in a variety of areas of advanced mathematics and science, as it provides a bridge between vectors and linear transformations.

Linear algebra is a very rich area with many uses within mathematics and many applications to science. For example, in computer science, Google's PageRank algorithm boils down to computing an eigenvector for the transition matrix of a certain Markov chain. As another example, data compression can be achieved using a low-rank matrix approximation which arises from the singular value decomposition of a matrix. In physics, linear algebra serves as a fundamental language of quantum mechanics. A strong grasp of the material in Math 3125 can offer great benefits when taking other courses and when pursuing a mathematical career.

Prerequisites: Math 2971 and 2185.

MATH 3257: INTRODUCTION TO COMPLEX VARIABLES

Complex analysis deals with the calculus of complex-valued functions  $f : \mathbb{C} \rightarrow \mathbb{C}$ . A real-valued function of a real variable maps an interval to another interval, but a complex-valued function of a complex variable maps a two-dimensional region to a two-dimensional region. To make sense of the derivative of  $f$ , it is natural to try to extend the definition of the derivative of a real-valued function to  $f(z) = u(x, y) + iv(x, y)$  where  $z = x + iy$ , i.e.,

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

Evaluating the limit first along the real and then the imaginary axis, we find  $f'(z) = u_x + iv_x$  and  $f'(z) = -iu_x + v_y$ , respectively. In order for  $f'(z)$  to be well-defined, we are led to the conditions:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{1}$$

Equations (1) are called the *Cauchy-Riemann equations*; they are foundational in complex analysis and lead to some surprising behavior of complex analytic functions.

A mapping  $z = f(\zeta)$  with  $f(\zeta)$  analytic and  $f'(\zeta) \neq 0$  preserves the angle between curves. Such a mapping is called a *conformal mapping* and can be used to solve problems in fluid flow, electrostatics, and other fields. Can you find an analytic function that conformally maps the interior of an ellipse to the unit disk? Or the upper-half complex plane to the exterior of the unit circle? Check out [Complex Analysis and Conformal Mappings](#) by Peter J. Olver or [Wolfram MathWorld](#).

The notion of line integral  $\int_C f(z)dz$  extends to a complex curve  $C \subset \mathbb{C}$  and a complex-valued function  $f(z)$ . The connection between analyticity and the complex integral is the content of the Cauchy-Goursat theorem: If  $C$  is a simple closed curve whose derivative is continuous except at a finite number of points and  $C$  is inside a simply connected region  $D$  in which  $f$  is analytic, then  $\int_C f(z)dz = 0$ . A converse of Cauchy's Theorem (Morera's Theorem) states that if  $\int_C f(z)dz = 0$  for a continuous  $f$  and every closed curve in  $R$ , then  $f$  is analytic in  $R$ . Cauchy's theorem can be used to prove the fundamental theorem of algebra, i.e., every polynomial of positive degree has at least one zero.

Another surprising result is presented by the Cauchy integral formula, which asserts that the values of an analytic function on the *boundary* of a disk determine its value at every *interior* point. If  $f(z)$  is analytic in a simply connected region  $D$  and  $C$  is a simple closed positively-oriented curve in  $D$  and  $z$  is a point inside  $C$ , then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw.$$

In fact, using complex integration and the Cauchy integral formula one can show that the values of an analytic function in its domain are completely determined by its values on an arbitrarily short curve in the domain.

The two-dimensional character of complex-valued functions is one of the reasons that complex analysis is so effective in two-dimensional problems in mathematical physics (hydrodynamics, thermodynamics, and quantum mechanics). In addition, complex analysis is useful in many branches of mathematics: algebraic geometry, number theory, combinatorics, partial differential equations, and applied mathematics.

Prerequisites: Math 2184 or 2185, Math 2233, and 2971.



## MATH 3342, ORDINARY DIFFERENTIAL EQUATIONS

Many principles that underlie the behavior of the natural world are relations that involve rates. When expressed in mathematical terms, the rates are derivatives and the relations are equations that involve derivatives, that is, *differential equations*. To understand real-world problems, whether it is classical or relativistic motion, turbulence or fluid dynamics, the shape of a drum or detection of seismic waves, epidemic spread of a virus, or financial market fluctuations, among many others, it is necessary to know differential equations and the mathematical models that they describe.

Math 3342 is a first course in differential equations, with an emphasis on mathematical modeling. Typically we start by discussing basic physical models, expressing them using ordinary differential equations, and exploring an initial approach to solutions via solution curves and direction fields. (“Ordinary” in “ordinary differential equations” refers to using functions of a single variable.) Then we classify types of differential equations and discuss what it means to solve a differential equation exactly or to get an approximate solution, and we discuss the issues of the existence and uniqueness of solutions.

The next major part focuses on first-order differential equations, which use only a function and its first derivative, not higher derivatives. We develop several methods to solve these equations exactly. This course relies on your knowledge of calculus, and, in particular, in this part you will need to use the integration techniques that you learned in *Calculus II*. It is not always possible to find exact solutions to differential equations, so we discuss numerical approximations of solutions, in particular, Euler’s method.

First-order differential equations are not sufficient to describe all physical processes. For example, Newton’s second law requires acceleration, which is the second derivative of displacement. Thus, we next discuss second-order differential equations. The approaches to study them generally differ from those for first-order differential equations. We start by discussing constant-coefficient linear second-order differential equations, their *characteristic equations*, and various possibilities for the roots (distinct or not, real or complex). We then discuss non-homogeneous equations, or processes with external forces, some applications in mechanics, and several approaches to obtain solutions.

It turns out that higher-order linear differential equations can be treated similarly to second-order ones, so we next generalize the approach from the second-order differential equations and then move on to basic *systems of differential equations*, since quite often in mathematical modeling we need to consider several changing functions, not just one. In this part of the course, a knowledge of matrices, eigenvalues, and eigenvectors from linear algebra is important.

We study mostly linear systems and learn how to understand the behavior of a system via *phase portraits* or *phase diagrams*. At the end of the course, we take a glance at nonlinear systems of equations. To understanding systems of nonlinear ordinary differential equations, we first find critical points, or equilibrium, steady-state solutions, then linearize a system around those critical points and, finally, try to create a basic phase portrait to understand the general behavior of a nonlinear system of differential equations, so it would be possible to interpret solutions for a given physical model.

Prerequisites: Math 2184 or 2185, and Math 2233.

## MATH 3343, PARTIAL DIFFERENTIAL EQUATIONS

Almost every theory in the physical world is built around a partial differential equation. There are the Navier-Stokes equation in fluid mechanics, the Maxwell equation in electromagnetism, the Schrödinger equation in quantum mechanics, and the Einstein equation in general relativity. In Finance the Black-Scholes equation evaluates the price of stock options; in Biology the Gierer-Meinhardt equation describes pattern formation and morphogenesis in cell development. Within Mathematics, analytic functions studied in Complex Analysis are just solutions of the Cauchy-Riemann equation; Soap films studied in Geometry are solutions of the minimal surface equation; in 2003 Grigori Perelman solved the Poincaré conjecture, a fundamental problem in Topology, using a partial differential equation called the Ricci flow.

Here is one of the equations you will encounter in this class. Consider a heat conducting rod of length  $L$  and we want to know its temperature. The temperature, denoted by  $u$ , is not a number but a field. It varies from point to point on the rod and also changes in time, so  $u$  is a function of the space variable  $x$  in the interval  $(0, L)$  and the time variable  $t > 0$ . Thermodynamics tells us that the temperature function  $u(x, t)$  satisfies the following heat equation:

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2}.$$

This equation is a partial differential equation because it involves some partial derivatives of the unknown variable  $u$ . To solve this problem, we also need to understand how the rod interacts with the external environment. As an interval in the real line, the rod meets the outside world at the two end points:  $x = 0$  and  $x = L$ . It is possible that the temperature is maintained at the fixed level, say  $A$  degrees at  $x = 0$  and  $B$  degrees at  $x = L$ . Then one imposes the boundary condition

$$u(0, t) = A, \quad u(L, t) = B, \quad \text{for all } t > 0.$$

It is also possible that the rod is insulated from the environment and there is no heat flow at the end points. Then one has a different boundary condition

$$\frac{\partial u(0, t)}{\partial x} = 0, \quad \frac{\partial u(L, t)}{\partial x} = 0, \quad \text{for all } t > 0.$$

Finally, we need to know the temperature distribution at the initial time, i.e.

$$u(x, 0) = \phi(x), \quad \text{for all } x \in (0, L)$$

where  $\phi$  is a given function called the initial condition. We will learn how to solve the heat equation for the unknown function  $u$  when proper boundary and initial conditions are provided. One of the main methods in this course is called separation of variables. It relies on the theory of trigonometric series in Advanced Calculus.

In the second half of the semester, we will consider bodies more complex than a rod. Say you have a heat conducting solid body, called  $D$ , in space. How does the temperature of the body evolve? Now we have an unknown function  $u(x, y, z, t)$  of  $(x, y, z)$  in  $D$ , which is a region in the three dimensional Euclidean space, and of time  $t > 0$ . The heat equation becomes

$$\frac{\partial u(x, y, z, t)}{\partial t} = \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2}$$

We must develop different approaches for different shapes of  $D$ .

Other equations studied in this course include the Laplace equation which is used to study gravity and static electric fields, the wave equation for sound, water, and electromagnetic waves, and the Schrödinger equation for quantum harmonic oscillators.

Prerequisites: Math 3342. You must know how to use the divergence theorem, how to integrate by parts in multiple integrals, and how to find eigenvalues in ordinary differential equations.

## MATH 3359, INTRODUCTION TO MATHEMATICAL MODELING

Mathematical modeling is the process of describing in mathematical terms some phenomenon or idea that is not initially perceived or described from a mathematical perspective. For example, one may try to develop equations to describe how some physical events are related, or a formal algorithm to capture how some decision is arrived at. Once this movement from the empirical world to the mathematical world has been carried out, one can bring a host of mathematical techniques to bear in an attempt to explain and/or predict aspects of the empirical world.

For instance, suppose a population of organisms is such that in an environment that provided unlimited resources, the population grows continuously, and at all times proportionally to its size  $u(t)$ . Writing  $a$  for the constant of proportionality, we can describe the population size by the differential equation  $\frac{du(t)}{dt} = au(t)$ . Now add the extra hypothesis that the organisms live in an environment that can only support up to a certain population size, say  $K$ , and that due to intra-species competition, the growth rate is inversely proportional to how close  $u(t)$  is to  $K$ . Under these assumptions, the population size can be described by the following more elaborate differential equation:  $\frac{du(t)}{dt} = au(t) \left(1 - \frac{u(t)}{K}\right)$ . This is called the “logistic equation,” and it has a very nice solution  $u(t)$ . But now suppose that the organisms in question have seasonal fertility, so that growth occurs in jumps. Keeping all the other assumptions the same while adjusting to use a sequence  $u_0, u_1, u_2, \dots$  rather than a continuous function  $u(t)$ , we can model this growth with a difference equation  $u_{n+1} = u_n + au_n \left(1 - \frac{u_n}{K}\right)$ . This is called the “discrete logistic equation,” and it does *not* have a very nice solution; in fact, it leads to what is called “chaotic behavior.” Constructing, studying, and contrasting examples of this kind is a key aspect of this course.

Mathematical modeling is a big field, and enjoys a variety of approaches, including well-known statistical and machine-learning techniques. In line with the above example, however, this course focuses primarily on “modeling from first principles.” This means that one begins by identifying key principles believed to be at play in the context to be modeled, expresses these principles in terms appropriate to some area of mathematics, then derives the logical implications, thereby developing an appropriate mathematical theory. Some elementary techniques of data-analysis are usually discussed, although they are not a primary focus.

The types of mathematical models considered in this course can be categorized according to two distinctions. Some models consider continuously varying quantities, such as temperatures and forces, and others consider discrete quantities, such as populations or packets of information. Some models are intended to describe dynamic systems, such as springs bouncing or economies growing, and others describe static systems, such as a balancing of forces. Different combinations of characteristics will typically involve applying material from different areas of mathematics. For example, the study of continuous dynamic systems often involves eigenvalue analysis of differential equations, whereas the study of continuous static systems may focus on minimizing some kind of energy functional. Discrete dynamical systems are sometimes modeled using iterated matrix multiplications and linear algebra techniques, whereas discrete static systems typically involve combinatorial graph theory.

Different instructors will choose to highlight different phenomena to model. Some commonly chosen topics include: spring systems, electrical circuits, population growth for single species, population growth for multiple species, and network flow. Mathematical techniques to be covered will typically include some or all of the following, and possibly others as well: dimensional analysis, minimization of linear and quadratic functionals in multiple variables, differential equations, eigenvalue analysis and decoupling of equations, analysis of periodic and other long-term behavior, difference equations, chaos, combinatorial graph theory, max-flow/min-cut theorem and matchings in combinatorial graphs.

Prerequisites: Math 3342 (and its prerequisite courses) and one of CSCI 1011, 1041, 1111, 1121 or 1131.

MATH 3410, MATHEMATICS OF FINANCE,  
MATH 3411, STOCHASTIC CALCULUS METHODS IN FINANCE

This pair of courses develops the mathematics and models used in the valuation of financial derivatives. Apart from the calculus and linear algebra prerequisites listed below, these courses are self-contained: the requisite probability theory is developed from first principles and introduced as needed, and finance theory is explained in detail without assuming prior familiarity with the subject.

We illustrate the basic idea of option valuation with a *call option*. This is a contract between two parties with the following conditions.

- At a prescribed time  $T$  the buyer (holder) of the option may (but need not) purchase a given stock for a prescribed amount  $K$ , established at the opening of the contract.
- At this time the seller (writer) of the option is required to sell the stock if the holder chooses to exercise the option.

If the option to buy is exercised, the holder earns the payoff  $(S_T - K)^+$ , where  $S_T$  is the value of the stock at time  $T$ . Since the holder has a right with no obligation, the option has a value and therefore a price, which must be paid to the writer by the holder at the time of opening of the contract. How does one establish a fair price for the option? The idea is to set up a dynamically adjusted portfolio consisting of the underlying stock and a bond, and use this to, in some sense, “match” the option value. The initial value of the portfolio is set so that the final value of the portfolio is precisely the payoff  $(S_T - K)^+$  to the holder. The law of one price then implies that the initial value of the portfolio is the price of the option.

Math 3410 develops option pricing in discrete time, using the binomial model to implement the general procedure described above. This model serves two purposes. First, it provides a way to price options using relatively elementary mathematical tools. Second, it allows a straightforward and concrete exposition of fundamental principles of finance, such as arbitrage and hedging, without the possible distraction of complex mathematical constructs. The course culminates in the establishment of the well-known Cox-Ross-Rubenstein formula for the price of an option.

Math 3411 develops option pricing in continuous time, which requires the methods of *stochastic calculus*. The basis of this calculus is the *Ito integral* and *Ito's formula*, the latter a fundamental tool in stochastic differential equations. Stochastic calculus methods are used to construct the Black-Scholes-Merton partial differential equation, the solution of which leads to the celebrated Black-Scholes formula for the price of a call option.

Prerequisites for Math 3410: Math 2233.

Prerequisites for Math 3411: Math 2184 or 2185; Math 3410.

## MATH 3553, INTRODUCTION TO NUMERICAL ANALYSIS

Numerical analysis is the study of algorithms that use numerical approximations (as opposed to general symbolic manipulations) for the problems of mathematical analysis. The overall goal of the field of numerical analysis is the design and analysis of techniques to give approximate but accurate solutions to hard problems. The field of numerical analysis includes many sub-disciplines. In Math 3553, we discuss some of the major ones.

- **Accuracy and precision:** The arithmetic performed by a calculator or computer is different from that in algebra and calculus courses. In our traditional mathematical world, we permit numbers with an infinite number of digits. In the computational world, however, each representable number has only a fixed, finite number of digits. The error that is produced when a calculator or computer is used to perform real number calculations is called round-off error. An algorithm is considered good when it keeps accuracy under computer arithmetic. We discuss some fundamental concepts such as round-off errors, computer arithmetic, convergence of algorithms etc.
- **Interpolation:** This is a method of constructing new data points within the range of a discrete set of known data points. In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate (i.e., estimate) the value of that function for an intermediate value of the independent variable. We will introduce the polynomial interpolation which can be represented in Lagrange or Newton form. We study the error estimate for such interpolation.
- **Solving nonlinear equations:** For a general function  $f(x)$ , one usually cannot solve  $f(x) = 0$  to find zeros analytically. On the other hand, we can use numerical methods to approximately solve for it. In this course, we will study bisection method and Newton method (fixed-point iteration method). The error analysis and the rate of convergence for such methods are discussed.
- **Numerical integration:** This constitutes a broad family of algorithms for calculating the numerical value of a definite integral. We will study the Newton-Cotes quadrature for uniform quadrature nodes. Then adaptive quadrature methods such as Gaussian quadrature are explored.
- **Direct Methods for linear systems:** The problem  $Ax = b$  arises in many areas of science and engineering. We introduce the Gauss elimination method, which induces LU decomposition, and Gauss elimination method with pivoting. For some matrices with special structures such as symmetric positive definite matrices, we study the Cholesky factorization method.
- **Iterative methods for linear systems:** These are usually more efficient than the direct methods. We discuss the Jacobi and Gauss-Siedel iterations and the SOR method. Conjugate gradient method are explored as well.
- **Approximation of functions:** These are discussed by exploring the least squares approximation, orthogonal functions such as Legendre polynomials, Chebyshev polynomials and Laguerre polynomials. We briefly discuss the rational function approximation.

Prerequisites: Math 2184 or 2185, and Math 2233, and one of CSCI 1011, 1041, 1111, 1121 or 1131.

MATH 3613, INTRODUCTION TO COMBINATORICS

Very roughly, combinatorics deals with finite objects and their structure. However, combinatorics is such a broad field that this description leaves much out. The main topic in Math 3613 is easier to nail down: the course focuses on questions of the type “How many . . . are there?”, as we illustrate below.

- (a) How many ways are there to cover all squares in an  $n$  by  $m$  grid with dominoes, where each domino covers two squares and no two dominoes cover the same square?
- (b) How many partitions of an  $n$ -element set into  $k$  subsets are there?
- (c) How many ways are there to dissect a regular  $(n + 2)$ -gon into triangles?
- (d) How many ways can we write a positive integer  $n$  as a sum of positive integers, written in non-increasing order? There are five for  $n = 4$ :  $4, 3 + 1, 2 + 2, 2 + 1 + 1$ , and  $1 + 1 + 1 + 1$ .

When we consider sequences of such problems, we often see structure that may not be apparent in an isolated instance. For instance, in problem (a) fixing  $n$  to be 2 and letting  $m$  vary, with  $m \geq 0$ , yields the sequence of answers starting with 1, 1, 2, 3, 5, 8, 13. If  $f_m$  denotes the general term, then  $f_m = f_{m-1} + f_{m-2}$ . Why? This recurrence relation reflects structure. In problem (b), let  $S(n, k)$  denote the number of partitions. If we fix  $n$  and let  $k$  vary from 0 to  $n$ , we get the polynomial identity

$$x^n = \sum_{k=0}^n S(n, k)x(x-1)(x-2)\cdots(x-k+1).$$

Why? Again, this algebraic relation reflects structure. This example hints at the fact that a key aspect of combinatorics is a powerful interplay with algebra. If  $C_n$  denotes the solution to problem (c) in case  $n$ , then we get the relation

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0.$$

Why? This recurrence relation reflects structure, and it is useful since it leads to an infinite series displaying  $C_n$ :

$$\frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 0} C_n x^n,$$

from which we can find  $C_n$ . The number  $p(n)$  that is the answer to problem (d) is easily expressed using a series and an infinite product:

$$\sum_{n \geq 0} p(n)x^n = \prod_{k \geq 1} \frac{1}{1 - x^k} = \frac{1}{1 - x} \cdot \frac{1}{1 - x^2} \cdot \frac{1}{1 - x^3} \cdot \frac{1}{1 - x^4} \cdots$$

Why? Again, algebra reflects important structure. We can learn much about  $p(n)$  through such representations. With the insight gained by studying combinatorics, we can look at identities such as

$$\prod_{i \geq 0} (1 + x^{2^{i+1}}) = \sum_{d \geq 0} \left( \prod_{i=1}^d \frac{1}{1 - x^{2^i}} \right) x^{d^2}$$

and get a conceptual understanding of them by seeing what each side counts.

Besides being inherently intriguing, the questions that combinatorics treats and the tools that it develops have applications in many fields, including probability, statistics, and computer science. Indeed, the historical roots of combinatorics include finding probabilities in games of chance (e.g., what is the probability of getting certain hand in a game of cards?).

All areas of combinatorics have a wealth of open problems. See

<http://www.openproblemgarden.org/category/combinatorics> and

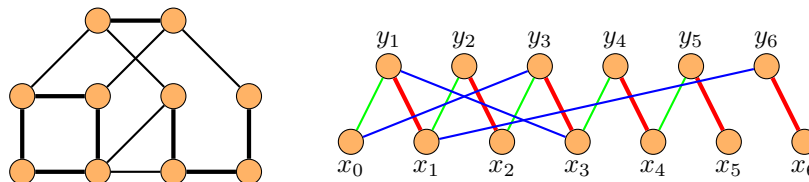
[https://en.wikipedia.org/wiki/List\\_of\\_unsolved\\_problems\\_in\\_mathematics](https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics).

For a fuller, yet still brief, description, see <https://en.wikipedia.org/wiki/Combinatorics>

Prerequisites: Math 2971.

## MATH 3632, INTRODUCTION TO GRAPH THEORY

It is often useful to depict data with diagrams made of dots and lines. For example, think of a family tree, an evolutionary tree, a diagram of administrative structure, a social network, the maps in airline magazines showing routes, or a network of roads. These are all examples of graphs, which we can represent as diagrams of vertices (dots) and edges. Applications are ubiquitous. For instance, to get a computer to play a game well, it is programmed to generate and analyze a graph in which the vertices are the possible states in the game and edges indicate which state can be reached from another in one move; the computer can then search this graph to find its optimal move.



Math 3632 focuses on theory, but the range of applications is enormous, and they motivate many of the topics considered.

If the graph on the left above represents committees, with an edge between vertices indicating that the committees share a member, how few time slots do we need in order to schedule meetings so that no person has a conflict? This motivates the graph coloring problem: what is the fewest number of colors needed so that we can assign colors to the vertices so that no vertices joined by an edge have the same color? Associated to a graph is a polynomial, its chromatic polynomial, whose value at each positive integer  $k$  gives you the number of valid colorings of the graph with a set of  $k$  colors.

If the graph on the right represents which applicants (the  $y$ s) are qualified for which jobs (the  $x$ s), what is the maximum number of jobs we can successfully fill? What is the maximum number of applicants who can get jobs? These are questions in matching theory. (The red edges in the illustration give a matching that satisfies all applicants.) Such questions are related to problems about selecting minimal sets of vertices for which every edge has an end-vertex in the set.

An electrical circuit is another example of a graph. If we want to print circuits on chips, it would be useful to know how few layers we need in order to print the circuit. In particular, can the circuit be drawn in the plane with no crossings (except where intended, at vertices)? This is the issue of planarity. For instance, the graph on the left above is not drawn in a planar manner, but it has a planar drawing: move the third vertex in the middle row below the bottom row, and replace the edge to its upper neighbor by an edge that curves around the left side of the graph. It turns out that there are two special graphs that completely control which graphs have planar drawings.

For a graph that depicts streets, you might ask for the number of different walks from your home to your favorite restaurant, no two of which use the same street. For a communications network, you might ask how many nodes can fail before messages can no longer be delivered among the remaining nodes? These are issues of graph connectivity. Results about connectivity are the work-horses behind many of the deepest results in graph theory.

Graph theory has a wealth of open problems. For instance, it is known that each planar graph can be drawn with edges that are straight lines, but it is still a conjecture that there is such a drawing in which the length of each edge is an integer (Harborth's conjecture). See

[http://www.openproblemgarden.org/category/graph\\_theory](http://www.openproblemgarden.org/category/graph_theory) and

[https://en.wikipedia.org/wiki/List\\_of\\_unsolved\\_problems\\_in\\_mathematics](https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics).

For a fuller, yet still brief, description, see [https://en.wikipedia.org/wiki/Graph\\_theory](https://en.wikipedia.org/wiki/Graph_theory)

Prerequisites: Math 2971.

## MATH 3710: INTRODUCTION TO MATHEMATICAL LOGIC

Mathematical logic is the study of reasoning using the exact methods of mathematics. It provides the foundations of mathematics and computer science. Its classical areas investigate mathematical structures in general (model theory), sets and infinities (set theory), and algorithmic processes (computability theory). One of its goals is to understand the degree to which reasoning can be formalized and mechanized.

Model theory provides a rigorous mathematical framework for the notions of language, provability, model, and truth. Model theory emerged as a distinct field in the 1940's through the works of Gödel, Löwenheim, Malcev, Skolem, Tarski, and others. A *model*, a concept used in all of the sciences, describes a portion of reality by using a formal language to express properties under study. In mathematics, every model can be reduced to a nonempty set of elements, called the *domain*, with certain operations and relations on these elements. For example, the *standard model of arithmetic* consists of the set of all natural numbers:  $0, 1, 2, 3, \dots$  with the binary operations of addition and multiplication.

We will focus on the first-order languages for theories and models, where formulas are built using constants, relation and functions symbols, variables (standing for the elements of the domain), Boolean operation symbols and universal and existential quantifiers ranging over variables. First-order languages are sufficient to express a great deal of current mathematical practice. A sentence is defined to be a formula with no free variables (that is, all variables are quantified), and a theory is defined to be any set of sentences. Every model has its (full) theory. For example, *full number theory* is the set of all first-order sentences true in the standard model of arithmetic.

One of the fundamental results in model theory is the *Gödel completeness theorem*, which establishes a correspondence between syntactic provability (derivability) and semantic truth using first-order language. Syntactic provability is based on the notion of a formal system given by a set of axioms and a set of logical rules of inference allowing consequences to be derived from axioms in a finite number of steps. Hence a formal system mimics the patterns of human reasoning, but establishes a set of rigorous rules for symbolic manipulation. A theory is consistent if no contradiction can be derived from it. The Gödel completeness theorem also establishes that a theory is consistent if and only if it has a model.

One of the consequences of the Gödel completeness theorem is the *compactness theorem*, which says that an infinite theory has a model if and only if every finite subtheory has a model. We will use the compactness theorem to show that there are nonstandard models of arithmetic, that is, models that satisfy the same sentences as the standard model of arithmetic, but also have infinitely large numbers.

Computability theory is a branch of logic that explores the power and limits of the algorithmic method. The *Gödel incompleteness theorems* are striking early results about its limits. The first Gödel incompleteness theorem establishes that no consistent algorithmic formal system that is sufficiently powerful proves every true statement. In particular, there is no consistent formal system with computable set of axioms that can capture all sentences true in the standard model of arithmetic. Peano developed a formal system with an infinite but computable set of axioms from which many sentences of full number theory can be derived (proved), but there are also sentences that are neither provable nor refutable. In this sense, Peano's system is incomplete. Moreover, Gödel showed that every such formal system is incomplete. Hence there are continuum many completions of Peano arithmetic, one of which is the full number theory.

Prerequisites: Math 2971.



## MATH 3720: AXIOMATIC SET THEORY

Set theory was founded at the beginning of the 20th century by George Cantor who invented infinite numbers. Cantor defined a set as a “collection into a whole of definite, distinct objects of our intuition or our thought”. However, this definition allows the existence of some unusual sets that lead to paradoxes. For example, the existence of the set of all sets leads to a paradox. These paradoxes show that there are properties that do not define sets, leaving set theorists with the task of determining which ones do define sets. Unfortunately, Gödel’s results imply that a complete answer to this question is not even possible. Therefore, axiomatic set theory attempts a less lofty goal. It formulates some of the fundamental properties of sets used by mathematicians as axioms. We will focus on Zermelo-Fraenkel axiomatic system. Within this axiomatic system practically all notions of contemporary mathematics can be defined and their properties can be derived. In this sense the axiomatic set theory serves as the foundation of mathematics.

Cantor proved that there are infinitely many infinite numbers. Some of these infinite numbers are called *cardinals* and they measure different infinite sizes. Every natural number is also a cardinal, measuring a finite size. The set of all natural numbers is countably infinite and has the smallest infinite size, denoted by  $\aleph_0$ . Cantor showed that, while there are as many integers as natural numbers and as many rational numbers as natural numbers, there are more real numbers than natural numbers. Is there an intermediate infinity? A negative answer to this question is known as the *continuum hypothesis*. It can be shown that there are as many complex numbers as real numbers, and as many infinite binary sequences as real numbers. In 1963, Paul Cohen obtained a surprising result, which was “rather unsatisfactory to an average mathematician,” by establishing that the continuum hypothesis is *independent*, that is, neither provable nor refutable from the usual set-theoretic axioms. The independence results use the *forcing technique* of building models for which Cohen won the Fields Medal.

Another well-known mathematical principle which the ordinary set theoretic axioms fail to settle is the *axiom of choice*. The axiom of choice states that given a family of nonempty sets, we can choose one element from each set. This statement is obvious if we have finitely many sets or some way of selecting distinguished elements in particular sets, but it is not possible to establish the statement in general for infinite families. However, the axiom of choice is used to prove many important results in analysis, topology, algebra and other areas of mathematics. What would happen with these results if we do not assume the axiom of choice? Do we need the full strength of the axiom of choice or would some weaker forms suffice? The use of the axiom of choice is often disguised since it is equivalent to hundreds of other mathematical statements. The most common ones are known as Zorn’s lemma or well-ordering principle. The well-ordering principle says that every set can be well-ordered. A set is well-ordered if every nonempty subset of the set has the least element.

Cardinals extend to more general numbers that are called *ordinals*, which correspond to well-orderings. There are so many ordinals that they cannot be collected into a set, as Burali-Forti paradox warns us. Yet we can pursue mathematical proofs by induction on ordinals. We can also justify definitions by transfinite recursion on ordinals. Among ordinals are the computable ones. They are isomorphic to computable well-orderings. The computable ordinals form an initial segment of the ordinals. The first non-computable ordinal is smaller than the first uncountable ordinal.

Prerequisites: Math 2971.

## MATH 3730: COMPUTABILITY THEORY

“Is there an algorithm that determines whether any given Diophantine equation has a solution in integers?” was the only decision problem among twenty-three problems posed by Hilbert at the Second International Congress of Mathematicians in 1900. It took seventy years to find its solution, which turned out to be negative, and three-and-half decades to establish computability theory needed for the solution.

*Computability theory* is a rigorous mathematical theory of algorithms or decision procedures. In the 1930's, Church, Gödel, Kleene, Post, Turing, and others developed computability theory as a precise mathematical theory of algorithms. Their results paved the way for the invention of a computer. One of the first results of computability theory was Turing's theorem that not all problems can be solved algorithmically; those that can be are called *decidable* or *computable*. A set is *computably enumerable* if it can be enumerated by an algorithm. Computable sets are those computably enumerable sets that are either finite or can be algorithmically enumerated in strictly increasing order. While computability theory demonstrated the power of algorithms, it also showed their intrinsic limitations. Building on the work of Davis, Putnam, and Robinson, Matiyasevich established in 1970 that every computably enumerable set can be realized as a Diophantine set. Hence a set of natural numbers is computably enumerable if and only if it is Diophantine. Furthermore, a set is computable if and only if both the set and its complement are computably enumerable. Not every computably enumerable set is computable; an example of such a set is Turing's halting set. These facts imply that Hilbert's Tenth Problem is undecidable.

Another striking limitation of the algorithmic method is described by the Gödel incompleteness theorems. These results, which strongly influenced all further thought about the foundations of mathematics, showed that Hilbert's dream of establishing the consistency of mathematics is impossible. The proof of the Gödel incompleteness theorems is based on syntactically representing all computable functions in the standard model of natural numbers with addition and multiplication.

Classical computability theory showed that most sets of natural numbers and problems they encode are not decidable (Turing computable). Only countably many functions on natural numbers can be computed algorithmically. One way to circumvent the limitations of standard algorithms is to consider generalized algorithms, which require oracles (a term introduced by Turing), that is, external knowledge to perform computations. The *Turing degree* provides an important measure of the level of such knowledge. All computable functions are of Turing degree zero because algorithms that work autonomously, requiring no oracle, can compute them. There are uncountably many Turing degrees and they are partially ordered. Turing degrees have the structure of an upper semilattice but not a lattice, and there are minimal degrees. An important countable substructure of this structure is the structure of computably enumerable degrees. These degrees also form an upper semilattice, but are dense (hence do not have minimal degrees). We will use Turing degrees and other computability-theoretic tools to investigate the complexity of problems in modern mathematics. While some mathematical constructions are algorithmic, or can be replaced by algorithmic ones yielding the same results, others are intrinsically non-algorithmic. An important example of negative result is the undecidability of the *word problem* in combinatorial group theory. The undecidability of Hilbert's Tenth Problem did not resolve the analogous decision problem for Diophantine equations with solutions in rational numbers. Hilbert's Tenth Problem for rationals is one of the main open problems in mathematics.

Prerequisites: Math 2971.

## MATH 3740: COMPUTATIONAL COMPLEXITY

By devising a conceptual machine that carried out algorithms, Alan Turing captured the essence of the intuitive notion of computability and developed the theoretical underpinnings of programmable digital computers. Since this and early achievements of computability theory in formalizing the concept of an algorithm, the theory of computation has developed into a broad and rich discipline. The study of *time* and *space* complexity measures of algorithms has resulted into an important area of mathematics and theoretical computer science known as computational complexity theory.

Computational complexity theory focuses on issues of computational efficiency, in an effort to explore the notion of what “real world” computers can do. It explores different ways of classifying problems in terms of computational resources required. By P we denote the class of problems that can be solved by a *deterministic* Turing machine in polynomial time, and by NP the class of problems that can be solved by a *nondeterministic* Turing machine in polynomial time. In contrast to the deterministic Turing machines, for which a computation is a single path of configurations, a computation of a nondeterministic Turing machine is a tree of configurations with multiple possible paths. *Polynomial time* (viewed as fast) means the time to complete the task varies as a polynomial function on the size of the input to the algorithm. The question whether P equals NP is one of the most important open problems in the theory of computation. Equivalently, it asks whether every problem the solution of which can be verified in polynomial time can also be solved in polynomial time. This problem is one of the seven millennium prize problems selected by the Clay Mathematics Institute in 2000.

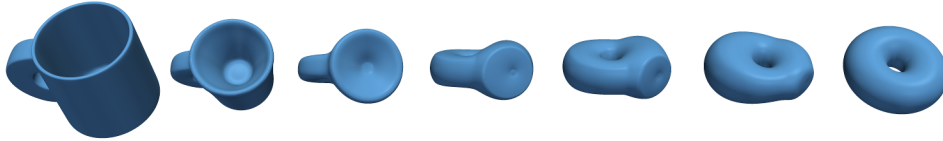
Consider the *traveling salesman problem*, an example of a problem that is fast to verify, but the answer to which may be slow to find. The traveling salesman problem asks the following question: “Given a list of cities and the distances between each pair of cities, what is the shortest possible path that visits each city and returns to the origin city?” A proposed solution is quickly verified as the time to check a solution grows polynomially as the number of cities gets bigger. However, all known algorithms for finding solutions take, for difficult examples, time that grows exponentially as the number of cities gets bigger. So the traveling salesman problem is in NP (quickly checkable) but we do not know whether it is in P (quickly solvable). Thousands of other problems are polynomially equi-reducible with the traveling salesman problem. They are called *NP-complete problems*. It can be shown that a fast solution to any one of these problems could be used to build a fast solution to all the others.

At present, with the size of the smallest circuit element approaching 100 atoms, we are reaching the limit of Moore’s Law for classical computation. However, a fascinating new theory of quantum computation and *quantum complexity theory* has been emerging. Quantum computers promise to be exponentially faster than the conventional ones of the same size. For example, it is not known whether a conventional computer can factor numbers fast; in fact, it is believed that this is an extremely time-consuming problem. In 1994, Peter Shor showed that a quantum algorithm can be used to factor numbers fast. Unlike conventional computers, which work by manipulating bits that exist either in state 0 or 1, quantum computers encode information as qubits, which exist in a superposition of both states. Like many quantum algorithms, Shor’s factoring algorithm is probabilistic: it gives the correct answer with high probability, and the probability of failure can be decreased by repeating the algorithm. An example of a deterministic quantum algorithm is the Deutsch-Jozsa algorithm, which always produces the correct answer.

Prerequisites: Math 2971.

## MATH 3806: INTRODUCTION TO TOPOLOGY

Topology is one of the younger branches of mathematics. It studies properties of geometric objects that are preserved under continuous deformations, such as stretching, crumpling, and bending, but not tearing or gluing. It is often said that a topologist cannot tell the difference between a coffee mug and a doughnut because one shape can be deformed into another in a few easy steps.



A topological space consists of a set and a collection of subsets, called open sets, that satisfy certain conditions. Two topological spaces are considered to be “the same” (*homeomorphic*) if there is a bijection between them that is continuous in both directions. For example, function  $\tan(\pi t/2)$  is a homeomorphism between an open interval  $(-1, 1)$  and the real line  $\mathbb{R}$ , while  $\mathbb{R}$  is homeomorphic to  $(0, \infty)$  via the exponential map. On the other hand,  $\mathbb{R}$  and the circle  $S^1$  are not homeomorphic since removing any point from the former always results in two pieces, while the latter remains in one piece. Any quantity or property that is preserved under homeomorphism is called a topological property. Important topological properties include connectedness and compactness. They are of great importance in real analysis, PDE, and other subjects.

In this course you will learn some of the major ideas and results in topology, as well as some recent questions, many of which remain unsolved. You will also see how various fields of mathematics, such as algebra, combinatorics, geometry, calculus, and differential equations, interact with topology.

One of the major results that we will prove is that all closed surfaces can be completely classified by computing a simple combinatorial invariant. Starting with a surface  $F$ , first put a few dots (called 0-cells or *vertices*) on  $F$  and then connect them by several non-intersecting segments (called 1-cells or *edges*) in such a way that the pieces that they split the surface  $F$  into all look like deformed disks (called 2-cells or *faces*). Let  $v$  be the number of vertices,  $e$  the number of edges, and  $f$  the number of faces. It turns out that the number  $\chi(F) = v - e + f$  does not depend on how the vertices and edges were chosen. It is called the *Euler characteristic* of  $F$ . It is easy to see that the 2-dimensional sphere  $S^2$  has Euler characteristic 2, while the (surface of a) doughnut (or a coffee mug) has Euler characteristic 0; thus, they are not homeomorphic. Surprisingly enough,  $\chi(F)$  can also be computed by considering vector fields on  $F$  and their critical points.

Other topological problems that we will consider include the following.

- **Fixed-point Theorems.** Given a continuous map of the disk to itself, some point must always remain fixed. It is not hard to convince yourself that this fact is plausible, but how can we prove it? What other spaces share this property? Fixed-point theorems have applications in numerous branches of mathematics, as well as physics, economics, and biology.
- **The Fundamental Theorem of Algebra.** Any polynomial  $t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$  of degree  $n \geq 1$  with complex coefficients has exactly  $n$  complex roots, counting multiplicity. The elegant topological proof of this theorem is one of the simplest.
- **Knots.** Take a piece of a string, interlace it in space, and then splice the ends together. When can two such knots be continuously deformed into each other? When can a knot be deformed into a trivial circle? The answers to these questions are surprisingly non-trivial and have applications in genetics, quantum physics, and chemistry. In fact, the study of knot theory was initiated in 1880’s by physicists and chemists who believed that atoms were merely knots in the ether, with different knots giving different elements.

Prerequisites: Math 2971.

## MATH 3848, DIFFERENTIAL GEOMETRY

Differential geometry explores the intrinsic geometry of curves and surfaces using methods from differential and integral calculus, and linear algebra. We begin with the study of planar curves and space curves parameterized by arc length, and continue onto the study of regular surfaces. This requires some knowledge of continuity and differentiability of functions and mappings in two and three-dimensional space. Roughly speaking, a regular surface is obtained by taking pieces of a plane and deforming them in such a way that there are no sharp points, rough edges, or self-intersections. The idea is to define a two-dimensional structure  $S$  where the tangent plane at  $p \in S$  is well-defined and we can extend the usual notions of differential calculus to functions and vector fields on  $S$ .

The rate of change of the direction of the tangent line to a curve  $C$  leads to the definition of the curvature of  $C$ . How quickly a surface pulls away from its tangent plane is measured using the unit normal  $N(p)$ , also called the Gauss map. The differential of the Gauss map at  $p$  (denoted  $dN_p$ ) is a linear self-adjoint map; its invariants are the Gauss curvature  $K$  and the mean curvature  $H$ . A surprising result is that  $K$  can be computed using only quantities related to the metric on  $S$  and its first derivatives. This result is closely related to Gauss's *Theorema Egregium*:  $K$  is invariant by local isometries. This means that if one bends a surface without stretching, the Gauss curvature  $K$  does not change. This has consequences for the cartographer in that *any* (flat) map of the earth must distort distances!

The generalization of a "straight line" to curved surfaces is the geodesic. For a regular surface, geodesics are (locally) the shortest path between points in  $S$ . You might be surprised by what is the shortest path from Dulles (IAD) to Seoul (ICN) (check out an app such as [DistanceCalculator](#)). Another surprise is that for a surface with curvature  $K$ , the interior angles  $\{\phi_1, \phi_2, \phi_3\}$  satisfy

$$\begin{aligned}\phi_1 + \phi_2 + \phi_3 &> \pi, & \text{when } K > 0 & \text{ (space positively curved),} \\ \phi_1 + \phi_2 + \phi_3 &= \pi, & \text{when } K = 0 & \text{ (flat space),} \\ \phi_1 + \phi_2 + \phi_3 &< \pi, & \text{when } K < 0 & \text{ (space negative curved).}\end{aligned}$$

This is related to one of the deepest theorems in differential geometry, the *Gauss-Bonnet Theorem*. A corollary of the Gauss-Bonnet Theorem states that for an orientable compact surface  $S$  with surface measure  $d\sigma$ ,

$$\int_S K d\sigma = 2\pi\chi(S)$$

where  $\chi(S) = 2 - 2g$  is the Euler-Poincaré characteristic of  $S$  and  $g$  is the number of handles of  $S$ .

The subject of differential geometry provides an excellent path to transition to higher mathematics in the study of ordinary and partial differential equations, the calculus of variations, complex analysis, topology, Riemannian geometry, and differentiable manifolds (a generalization of the notion of a regular surface). The list of applications of differential geometry continues to grow, ranging from problems in machinery design, to the classification of four-manifolds, to the fundamental forces in nature, and to the study of DNA.

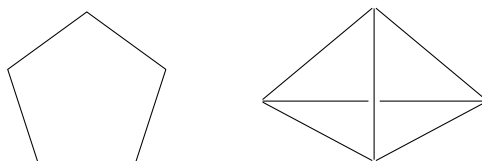
Prerequisites: Math 2184 or 2185; Math 2233 and Math 2971.

## MATH 4121 AND 4122, INTRODUCTION TO ABSTRACT ALGEBRA, I AND II

In your mathematics education, you first learned about numbers and operations on them, and their properties (for example, addition is associative, as is multiplication, and multiplication distributes over addition). Later you learned about polynomials and operations on them, and you saw that many of the same properties hold, including the three just cited. Abstract algebra is the general study of such systems. In abstract algebra, we study sets and operations on those sets that are subject to certain rules, such as those cited above. We work in an abstract setting; we illustrate the theory with, and apply it to, particular examples, but our interest is in proving results that hold in all examples. The proofs we give in the general setting apply to all examples, which makes for great efficiency. It also gives us insight into what holds in general versus what depends on the details of a given example.

Rings are a class of objects studied in abstract algebra that include many familiar examples. Rings have operations of addition and multiplication defined on a set, subject to familiar rules, except that we do not require non-zero elements to have multiplicative inverses. An example you know is the set  $\mathbb{Z}$  of integers with the usual operations; 2 has no multiplicative inverse since  $1/2 \notin \mathbb{Z}$ . In general rings, multiplication might not commute, so we can also consider rings of  $n$  by  $n$  matrices (matrix multiplication in general does not commute).

We also study less constrained structures, specifically, structures with just one operation. In the structure called a group, we have a set, one operation, and that operation is associative, has an identity, and each element has an inverse. Symmetry groups are one of the many important examples of groups. For instance, in a geometric figure, such as a regular pentagon or a tetrahedron, we can take two symmetries and follow one by the other (as in function composition). For instance, we can rotate the tetrahedron by  $120^\circ$  about the line that passes through a vertex and the center of the opposite face, and follow that by a  $180^\circ$  rotation about the line that passes through the centers of opposite sides. Each such symmetry has an inverse (in the first case, rotate another  $240^\circ$  about the same line; in the second case, apply the same rotation again). Another familiar example of a group is the collection of subsets of a fixed set with the operation of symmetric difference:  $A \triangle B = (A \cup B) - (A \cap B)$ . In that example,  $\emptyset$  is the identity and each set is its own inverse.



Math 4122 often treats two very striking examples of using algebra to solve classical problems: the impossibility of trisecting general angles with a ruler and compass, and proving that there cannot be counterparts of the quadratic equation for polynomials of degrees five or higher.

Abstract algebra is used in many other fields. It has very strong connections with number theory. In topology, one associates groups with topological spaces with the aim of distinguishing between the spaces by distinguishing between the associated groups. Abstract algebra has many important applications to a wide variety of other areas, including cryptography, coding theory, and physics.

Abstract algebra has many classical and new open problems. Much research now focuses on the interface between algebra and other fields. See

<http://www.openproblemgarden.org/category/algebra>

[http://www.openproblemgarden.org/category/group\\_theory](http://www.openproblemgarden.org/category/group_theory) and

[https://en.wikipedia.org/wiki/List\\_of\\_unsolved\\_problems\\_in\\_mathematics](https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics).

For a fuller, yet still brief, description, see [https://en.wikipedia.org/wiki/Abstract\\_algebra](https://en.wikipedia.org/wiki/Abstract_algebra)

Prerequisites for Math 4121: Math 2971 and either Math 2184 or 2185.

Prerequisites for Math 4122: Math 4121.

## MATH 4239 AND 4240, REAL ANALYSIS, I AND II

The word analysis in mathematics refers to the study of functions and their limits. Hence real analysis is the study of functions of a real variable or of several real variables. The first college-level course in analysis is calculus 1 and 2, the familiar first-year course about differentiation and integration of functions of a single real variable.

Math 4239, the first semester of the real analysis sequence, revisits the same ideas as are discussed in calculus 1 and 2: limits, continuity, differentiation, integration, sequences, and series. The course is sometimes called *Advanced Calculus*. (In this context, we refer to the calculus 1 and 2 sequence as *Elementary Calculus*.) Even though the topics in advanced calculus seem to be the same as in elementary calculus, the perspective is so different that it sometimes seems to students that they are entirely different subjects. In the elementary courses, almost every problem is about some formula, like  $x \sin x$  or  $e^{-x^2}$  or  $\arctan(\ln(x+1))$ , and the goal is to execute some computation with that formula. In Math 4239, hardly any problems ask specifically about one formula; instead, you learn what must be true about *any* formula. So in the elementary calculus sequence, a typical problem might begin “Suppose  $f(x) = x^2 + \tan(x)$ .” In Math 4239, a typical problem might begin “Suppose  $f$  is a continuous function from  $\mathbb{R}$  to  $\mathbb{R}$ .” This is the abstract viewpoint, which we adopt because it is efficient and enlightening.

Math 4239 investigates the foundations behind what is taught in calculus 1 and 2. We investigate various conditions that ensure the existence of limits, derivatives, and integrals. In calculus 1 and 2, we start with nice formulas that tend to automatically guarantee the existence of such things, and we simply compute them. The focus of Math 4239 is the assortment of theorems that assert such guarantees. So real analysis is the theory behind calculus.

Here are a few questions that illustrate ideas found in Math 4239:

- Can a function from  $\mathbb{R}$  to  $\mathbb{R}$  be continuous at all  $x$  but differentiable at no  $x$ ?
- Can a function from  $\mathbb{R}$  to  $\mathbb{R}$  be continuous at every irrational number but continuous at every rational number?
- Does every continuous function on  $[0, 1]$  have a definite integral?
- Under what circumstances does a Maclaurin series represent the function from which it came?

To analyze such questions, certain concepts are introduced that typically are not seen in the elementary calculus sequence. These concepts include open and closed sets, completeness, compactness, liminf and limsup, bounded variation, uniform continuity, and uniform convergence.

Part II of the course, Math 4240, covers the theory behind multivariable calculus. We develop an understanding of the derivative as a linear operator, with the familiar chain rule manifesting itself as a theorem about the product of matrices. We rework the definition of the Riemann integral in more than one dimension. We state, prove, and use the implicit and inverse function theorems. And if time permits, we may introduce the idea of a differential form in order to understand what is called the generalized Stokes theorem, which is a grand generalization and unification of the fundamental theorem of calculus, the fundamental theorem of line integration, Green’s theorem, Gauss’s theorem, and Stokes’s theorem.

The real analysis courses sit centrally between pure and applied mathematics. They display the abstraction and rigor of pure mathematics, but the objects of study are at the center of all applied mathematics. Every student contemplating post-graduate study in mathematics (pure or applied), physics, or economics should complete this sequence, or at the very least the first half of it.

Prerequisites for Math 4239: Math 1232 and Math 2971.

Prerequisites for Math 4240: Math 2184 or 2185, Math 2233, and Math 4239.

MATH 4981, MATH 4995, COURSES OUTSIDE THE DEPARTMENT, AND GRADUATE COURSES

- **Math 4981, Seminar: Topics in Mathematics.** The topic of Math 4981 changes each time it is offered, according to the interests of the instructor. Recent topics have included cryptography, the mathematics of medical imaging, and mathematical biology. Topics are announced shortly before the relevant registration period, via the list server for math majors and minors. Since the topic changes, this course can be taken multiple times.
- **Math 4995, Reading and Research.** This course is taken mainly by students who are working on senior honors theses (see the [Special Honors](#) description in the Bulletin). Occasionally it is also used to pursue a topic that is not covered in our regular courses. Note: the agreement of an instructor to supervise the course is required **before** a student signs up for Math 4995. Math 4995 can be supervised by any regular faculty member (not just the person listed in the schedule of classes).
- **Courses in other departments.** There are a few courses outside of the Math Department that we recognize as acceptable electives (typically at most one per student): Stat 4157, Stat 4158, Stat 4189, Stat 4190. If you are interested in taking one of these courses as an elective, talk with an advisor first, and, once you have taken the course, if it does not automatically show up as an elective in Degree Map, please remind the advisor for math majors and minors to submit a Degree Map petition.
- **Graduate courses.** Courses numbered 6000 and higher are intended for graduate students, not undergraduates. Some of our most advanced seniors who are preparing for graduate study in mathematics take one or two carefully-selected graduate courses in their senior year, but this must be done only after consulting the course instructor and an advisor for the major.